Function of a several variables

Q. What is a function?

Function is a rule

Ex: $y = 2x + 3$

dependent → independent variable

→ Role of dependent and independent variable

Dependent variable → Range

Independent variable → Domain, Dimension, Path

$y = f(x)$

1. Domain : x-axis
2. Range : y-axis

$z = f(x, y)$

1. Two independent variable
2. Two dimension
3. Domain : xy plane
4. Range : z-axis

* Range is always ⊥ to domain.

Partial Differential Equation

1. $y = f(x)$

$y' = f'(x)$

Nature of the path of derivative

2. If the function have a single independent variable, function can be expressed in ordinary derivative.

1. $z = f(x, y)$

If the function have more than one independent variable function can be expressed in parteal derivative.

2. Notation

(a) $\left.\dfrac{\partial z}{\partial x}\right|_{y \to \text{constant}}$

(b) $\left.\dfrac{\partial z}{\partial y}\right|_{x \to \text{constant}}$

(c) $\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{y \to \text{constant}}$

(d) $\left.\dfrac{\partial^2 z}{\partial y^2}\right|_{x \to \text{constant}}$

(e) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Q $z = \sin x \cos y$

$\frac{\partial z}{\partial x} = \frac{\partial(\sin x \cos y)}{\partial x} = \cos x \cos y$

$\frac{\partial z}{\partial y} = \sin x \frac{\partial(\cos y)}{\partial y} = -\sin x \sin y$

$\frac{\partial^2 z}{\partial y^2} = -\sin x \cos y$

→ Limit : Function of two variables

↳ Approximation

$f(x,y) = \frac{x^2-y^2}{x-y}$; this function is defined ∀ the values of x and y except $x=0$ and $y=0$, $x=y$.

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y}$ | $x>0, y>0$ / $x<0, y<0$

$\lim_{x\to x_0} f(x) = L \to$ fixed and unique

$x \to x_0 \to$ fixed and unique

$f(x) = \frac{x^2-2^2}{x-2}$

$\lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = 4$

∴ $\lim_{x\to 2^+} x+2 = 4 \to$ Fixed

↳ Fixed

A function $f(x,y)$ have limit L, if for every $\epsilon > 0$, $\exists$ a positive number $\delta > 0$, such that

$$|f(x,y) - L| < \epsilon, \text{ whenever } \begin{array}{l} |x - x_0| < \delta \\ |y - y_0| < \delta \end{array}$$

Symbolically,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

→ Independent Path: Choose any path function value must be same.

Example 1: Find the limit of function $f(x,y) = \dfrac{xy}{x^2+y^2}$ where $x \to 0$ and $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

→ First Method

Choose a path $y = mx$

$$\therefore \lim_{(x,mx)\to(0,0)} \frac{x(mx)}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

As, m is constant => Limit does not exist.

Example 2:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

Choose a path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x-mx}{x+mx} = \frac{1-m}{1+m}$$

; Here m is constant.

As, function depends on constant. Hence, the limit does not exist.

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Example 3:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - x\sqrt{y}}{x^2+y}$$

Choose a path $y = mx^2$

$$\lim_{(x,mx^2)\to(0,0)} \frac{x^2 - x\sqrt{mx^2}}{x^2+mx^2} = \lim_{(x,mx^2)\to(0,0)} \frac{\cancel{x^2}(1-\sqrt{m})}{\cancel{x^2}(1+m)} \text{ ; } m \text{ is constant}$$

As; function is dependent on constant 'm'.

$\Rightarrow$ Limit does not exist.

Example 4:

$$\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right) = \pm\frac{\pi}{2} \left[\begin{array}{c}\text{Not fixed}\\ \text{and not}\\ \text{unique}\end{array}\right]$$

Example 5:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$$

Choose a path $y = mx$

$$\lim_{(x,mx)\to(0,0)} \frac{x^3(mx)}{x^6+(mx)^2} = \frac{mx^4}{x^2(x^3+m)} = \frac{mx^2}{x^3+m} = 0.$$

Hence, limit exists.

But, if we choose path $y = mx^3$

$$\therefore \lim_{(x,mx^3)\to(0,0)} \frac{x^3\, mx^3}{x^6+(mx^3)^2}$$

$$\lim_{x\to(0,0)} \frac{mx^6}{x^6(1+m^2)} = \frac{m}{1+m^2}$$

As, function is dependent on constant

$\Rightarrow$ Limit doesnot exist.

Example 7

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+y^2} = \lim_{y\to 0} \lim_{x\to 0} \frac{e^{xy}}{x^2+y^2}$$

$$\lim_{y\to 0} \frac{1}{y^2} = \infty$$

Hence, Limit does not exist.

Continuity of function of two variables;

A function (x,y) is said to be continuous at a point (x_0, y_0), if

1. function $f(x,y)$ defined at $x=x_0$ and $y=y_0$.
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

If function $f(x,y)$ is defined in domain D. Let $f(x_0,y_0)$ is also defined in D. Then function $f(x,y)$ is continuous at given point (x_0,y_0) if for given $\epsilon > 0$ however small there exist a positive number $\delta > 0$ such that.

$$|f(x,y) - f(x_0,y_0)| < \epsilon, \text{ whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Example: Test continuity of

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the point $(0,0)$

By the definition

1. Function defined at (x_0, y_0)

Here the given function $f(x,y)$ is defined at the point $x_0 = 0$; $y_0 = 0$. The value of the function at the point $x = 0$, $y = 0$ is equal to 0.

2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2}$$

Choose a path $y = mx$

$$\Rightarrow \lim_{(x,mx)\to(0,0)} \frac{x^2 m}{x^2(1+2m^2)} = \frac{m}{1+2m^2}$$

As, function is dependent on constant 'm'.
Hence, Limit does not exist.

Test the continuity of

$$f(x) = \begin{cases} \dfrac{x-y}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the point $(0,0)$.

By the definition of continuity :

1. Function defined at (x_0, y_0)

$\because$ Function is defined at a point $x_0 = 0$ and $y_0 = 0$, the value of the function $f(x,y)$ at the point $x_0 = 0, y_0 = 0$ is equal to 0.

2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

$$\because \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

$\therefore$ Choose a path $y = mx$

$$\lim_{(x,mx)\to(0,0)} \frac{x(1-m)}{x(1+m)} = \frac{1-m}{1+m}$$

As, function is dependent on constant

$\Rightarrow$ Hence, Limit does not exist.

3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

As, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = 0$$

and $\frac{1-m}{1+m} \neq 0$

$\therefore$ Function is discontinuous at point $(0,0)$.

Example ; Test the continuity of the function

$$f(x) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

By the definition of continuity:

1. Function defined at (x_0, y_0)

$\therefore$ $f(x,y)$ is defined at a point $x=0$ and $y=0$, the value of the $f(x,y)$ at the point $x_0 = 0$, $y_0 = 0$ is equal to 0.

2. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - x\sqrt{y}}{x^2 + y^2}$$

Choose a path: $y = mx^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - x^2\sqrt{m}}{x^2 + (mx^2)^2} = 1 - \sqrt{m}$$

As, function depends on 'm'

Hence, Limit does not exist.

3. $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

$\Rightarrow$ As, $1 - \sqrt{m} \neq 0$

$\Rightarrow$ Function is discontinuous at $x_0 = 0$ and $y_0 = 0$.

Partial Differentiation

→ Homogenous Function : A function in which each term have a same degree.

$$z = f(x,y) = x^n + x^{n-1}y + x^{n-2}y^2 + \dots + xy^{n-1} + y^n.$$

Degree 8 n

$$x^n\left[x + \frac{y}{x} + \left[\frac{y}{x}\right]^2 + \dots + \left[\frac{y}{x}\right]^{n-1} + \left[\frac{y}{x}\right]^n\right]$$

$$z = x^n f\left(\frac{y}{x}\right).$$ Homogenous function whose degree is n.

Ex 1 : Find the degree

$$f(x,y) = \frac{x^3 - y^3}{x^2 + y^2}$$

$$f(x,y) = \frac{x^3[1 - (y/x)^3]}{x^2[1 + (y/x)^2]}$$

$$z = x\, f\left(\frac{y}{x}\right)$$

$\Rightarrow$ $n = 1$. Hence, degree $= 1$.

Ex 2 : Find the degree

$$z = f(x,y) = \frac{x^{5/2} - y^{5/2}}{x^{1/2} + y^{1/2}}$$

$$z = \frac{x^{5/2}(1 - (y/x)^{5/2})}{x^{1/2}(1 + (y/x)^{1/2})}$$

$$z = x^2 f\left(\frac{y}{x}\right)$$

$\Rightarrow$ $n = 2$. Hence degree $= 2$.

Ex 3 : Find the degree

$$z = y \tan^{-1}\frac{y}{x}$$

$$= \frac{x}{x}y\tan^{-1}\left(\frac{y}{x}\right) = x f\left(\frac{y}{x}\right) \Rightarrow n = 1.$$

Ex : Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at the path $(0,0)$ but not partially differentiable at point $(0,0)$.

By the definition of continuity

$$|f(x,y) - f(x_0,y_0)| < \epsilon, \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

As, $f(x,y) = x+y \sin\left[\frac{1}{(x+y)}\right]$

and $f(0,0) = 0$

$$\Rightarrow |f(x,y) - f(0,0)| = \left|(x+y)\sin\left(\frac{1}{x+y}\right) - 0\right|$$

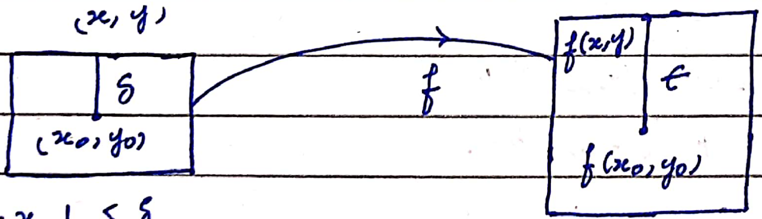
$$= |(x+y)|\,\cancel{\left|\sin\frac{1}{(x+y)}\right|} \to 1$$

$$\Rightarrow |x+y| \leq |x| + |y| \leq 2\sqrt{x^2+y^2} < \epsilon$$

Hence, the given function is continuous at the point $(0,0)$

$$\left|(x+y)\sin\frac{1}{(x+y)} - f(0,0)\right| < \epsilon \text{ whenever}$$

$$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$$



$|x - x_0| < \delta$

$|y - y_0| < \delta$

$$|f(x,y) - f(x_0,y_0)| < \epsilon$$

By the definition of the partial differentiation along x-axis :

$$f(x,y) \xrightarrow{\Delta x} f(x+\Delta x, y)$$

$$\left.\frac{\partial z}{\partial x}\right|_{y=\text{cons}} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{y=\text{cons}} = f_x(x,y)$$

$$\Delta z = f(x+\Delta x, y) - f(x,y)$$

$$\frac{\Delta z}{\Delta x} = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$\therefore$ $f(x+\Delta x, y) - f(x,y) = (x+\Delta x+y)\sin\left[\frac{1}{x+\Delta x+y}\right] - (x+y)\sin\left(\frac{1}{x+y}\right)$

$\therefore$ At $x=0$ and $y=0$

$\Rightarrow$ $f(x+\Delta x, y) - \cancel{f(0,0)}^{\,0} = \Delta x \sin\frac{1}{\Delta x} - 0$

$\therefore$ $f(\Delta x) = \Delta x \sin\frac{1}{\Delta x}$

$$\lim_{\Delta x \to 0} f(\Delta x) = \lim_{\Delta x \to 0} \Delta x \sin\frac{1}{\Delta x}$$

$$\lim_{\Delta x \to 0} f(\Delta x) = \begin{cases} +1, & \Delta x > 0 \\ -1, & \Delta x < 0 \end{cases}$$

$\therefore$ As, Limit is not ~~continuous~~ unique.

Hence, Limit does not exist.

$\therefore$ $f(x,y)$ is not partially differentiable along x-axis at $(0,0)$.

Ex; Show that the function,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

not continuous at $(0,0)$ but partially differentiable $(0,0)$.

By, the definition of continuity
Choose a path
$y = mx$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} = \lim_{(x,mx)\to(0,0)} \frac{mx^2}{x^2(1+2m^2)} = \frac{m}{1+2m^2}$$

As, function is dependent on m
$\Rightarrow$ Hence, limit does not exist.

$$f(x,y) \xrightarrow{\Delta x} f(x+\Delta x, y)$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{y\to\text{constant}} = \frac{\partial f(x,y)}{\partial x} = f_x(x,y)$$

$$\Rightarrow \lim_{\Delta x\to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\therefore \frac{\partial z}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\Rightarrow f(x+\Delta x, y) - f(x,y) = \frac{(x+\Delta x)y}{(x+\Delta x)^2+2y^2} - \frac{xy}{x^2+2y^2}$$

At point $(0,0)$

$$\therefore f(0+\Delta x, 0) - f(0,0) = \frac{(\Delta x)0}{(\Delta x)^2} - 0$$

$$\Rightarrow \lim_{\Delta x\to 0} f(\Delta x) = 0$$

$\therefore$ As, Limit is fixed and unique
$\Rightarrow$ Limit exists

Similarly, with y-axis
$\Rightarrow$ $f(x,y)$ is partially differentiable along x-axis as well as y-axis at point $(0,0)$.

→ Euler Homogenous Theorem

Statement: If $f(x,y)$ is homogenous function in x and y of degree m, then

*** $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

→ Proof: Consider a homogenous function

$$Z = x^n f\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

Partially differentiate equation (1) w.r.t x, where y is a constant.

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial\left(x^n f\left(\frac{y}{x}\right)\right)}{\partial x}$$

$$\frac{\partial z}{\partial x} = x^n \frac{\partial(f(y/x))}{\partial x} + f\left(\frac{y}{x}\right)\frac{\partial x^n}{\partial x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = x^n \frac{\partial\left(f\left(\frac{y}{x}\right)\right)}{\partial\left(\frac{y}{x}\right)} \cdot \frac{\partial\left(\frac{y}{x}\right)}{\partial x} + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial z}{\partial x} = -yx^{n-1} f'\left(\frac{y}{x}\right) + nx^{n+0} f\left(\frac{y}{x}\right) \quad \text{--- (2)}$$

Again, partially differentiating equation (1) w.r.t y, where x is constant

$$\therefore \frac{\partial z}{\partial y} = x^n \frac{\partial f(y/x)}{\partial(y/x)} \cdot \frac{\partial(y/x)}{\partial y} + \cancel{f\left(\frac{y}{x}\right)\frac{\partial x^n}{\partial y}}^{\,0}$$

$$y\frac{\partial z}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right) \quad \text{--- (3)}$$

On adding ② and ③ and on using ①,
We get :

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Hence proved.

→ Total differentiation

If $z = f(x, y)$

and

$$x = \phi(t)$$
$$y = \psi(t)$$

then

$$z = f(\phi(t), \psi(t))$$

$$\therefore \frac{\partial z}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

→ Composite Function (Chain Rule)

If $z = f(x, y)$

where

$$x = \phi(u, v)$$
$$y = \psi(u, v)$$

then

$$z = f(\phi(u,v), \psi(u,v))$$

$\frac{\partial z}{\partial u}$; where v is constant

$$f \to x \to u$$
$$f \to y \to u$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

For : $\frac{\partial z}{\partial v}$; where u is constant

$$f \to x \to v$$
$$f \to y \to v$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Q. If $x = e^{2u} + e^{-2v}$

$y = e^{-2u} + e^{2v}$

then prove that

$$\frac{\partial y}{\partial u} - \frac{\partial f}{\partial v} = z\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$$

By using Chain Rule :

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial u} = \frac{\partial(e^{2u}+e^{-2v})}{\partial u} = 2e^{2u}$$

$$\frac{\partial x}{\partial v} = \frac{\partial(e^{2u}+e^{-2v})}{\partial v} = -2e^{-2v}$$

$$\frac{\partial y}{\partial u} = \frac{\partial(e^{-2u}+e^{2v})}{\partial u} = -2e^{-2u}$$

$$\frac{\partial y}{\partial v} = 2e^{2v}$$

$$\therefore x\frac{\partial z}{\partial u} = \left\{\frac{\partial f}{\partial x}\cdot 2e^{2u} + \frac{\partial f}{\partial y}\cdot(-2e^{-2u})\right\}(e^{2u}+e^{-2v})$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot(-2e^{-2v}) + \frac{\partial f}{\partial y}\,2e^{2v}$$

$$\therefore \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = (2e^{2u}+2e^{-2v})\frac{\partial f}{\partial x} + (-2e^{2u}-2e^{2v})\frac{\partial f}{\partial y}$$

$$= 2\left\{x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right\}$$

→ Deduction of Euler's formula

If $z = f(x, y)$

and $f(x, y)$ is a homogenous function with degree 'n', then

$$x \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial z}{\partial x \partial y} + y \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Q. If $u = \cos^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then find

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

$\therefore \quad u = \cos^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$\Rightarrow \quad \frac{x^2(1+(y/x)^2)}{x(1+(y/x))} = \cos u$

$\Rightarrow \quad$ Let $z = \cos u = x f(y/x)$

Comparing with Homogenous equation

$\Rightarrow \quad n = 1.$

Hence,

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = nz$$

. By using Chain Rule and Euler Homogenous Theorem:

$z \longrightarrow u \longrightarrow x$

$z \longrightarrow u \longrightarrow y$

$$\therefore \quad x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = 1.z$$

$$x.\frac{\partial(\cos u)}{\partial u}.\frac{\partial u}{\partial x} + y\frac{\partial(\cos u)}{\partial u}\frac{\partial u}{\partial y} = \cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\cot u \text{ . ans}$$

Q. If $u = \tan^{-1}\left[\frac{x^{3/2}+y^{3/2}}{x^{1/2}+y^{1/2}}\right]$. then find

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

$\therefore \quad u = \tan^{-1}\left[\frac{x^{3/2}+y^{3/2}}{x^{1/2}+y^{1/2}}\right]$

$\therefore \quad \tan u = \frac{x^{3/2}+y^{3/2}}{x^{1/2}+y^{1/2}}$

Let $z = \tan u$

$\therefore$ Let $z = \dfrac{x^{3/2}\left[1 + (y/x)^{3/2}\right]}{x^{1/2}\left[1 + (y/x)^{1/2}\right]}$

$$z = x\left[\frac{1 + (y/x)^{3/2}}{1 + (y/x)^{1/2}}\right]$$

$$z = x\, f\left(\frac{y}{x}\right)$$

Comparing with $z = x^n f\left(\frac{y}{x}\right)$

$n = 1.$

Using Euler's Homogenous Equation ;

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial(\tan u)}{\partial u} \cdot \frac{\partial u}{\partial x} + y\frac{\partial(\tan u)}{\partial u} \cdot \frac{\partial u}{\partial y} = 1 \times \tan u$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \sin u \cos u$$

Ans.

Q. If $z = \sin^{-1}\left(\frac{x}{y}\right) + \sin^{-1}\left(\frac{y}{x}\right)$, then find

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ?$$

As ; $z = \sin^{-1}\left(\frac{x}{y}\right) + \sin^{-1}\left(\frac{y}{x}\right)$

$$z = y^0 \sin^{-1}\left(\frac{x}{y}\right) + x^0 \sin^{-1}\left(\frac{y}{x}\right)$$

Hence ; $n = 0.$

Using Euler's Homogenous Formula;

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n z$$

As; $n = 0$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0. \quad \text{ans}$$

→ Jacobian (J)

Q. What is the application of the Jacobian?

Ans. Transformation between plane.

$$\boxed{xy\text{-plane}} \xrightarrow{J} \boxed{uv\text{-plane}}$$

Q. How will you construct Jacobian?

With the help of a chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Proof using Crammer's Rule;

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Q. If $x = u(1+v)$

and $y = v(1+u)$

Then find J

$$J = \frac{\partial(x, y)}{\partial(u, v)}$$

$$\therefore \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\therefore \quad \frac{\partial x}{\partial u} = 1 + v \qquad \frac{\partial x}{\partial v} = 0 + u$$

$$\frac{\partial y}{\partial u} = 0 + v \qquad \frac{\partial y}{\partial v} = 1 + u$$

$$\therefore \quad J = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v + \cancel{uv} - \cancel{uv}$$

$$J = 1 + u + v$$. ans

Q. If $x = a\cosh\alpha\cos\beta$
$y = a\sinh\alpha\sin\beta$
then find J

$\cosh^2\alpha - \sinh^2\alpha = 1$

$\therefore \quad J = \dfrac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial x}{\partial\beta} \\ \dfrac{\partial y}{\partial\alpha} & \dfrac{\partial y}{\partial\beta} \end{vmatrix}$

$\therefore \quad \dfrac{\partial x}{\partial\alpha} = \dfrac{\partial(a\cosh\alpha\cos\beta)}{\partial\alpha} = a\sinh\alpha\cos\beta$

$\dfrac{\partial x}{\partial\beta} = -a\cosh\alpha\sin\beta$

$\dfrac{\partial y}{\partial\alpha} = a\cosh\alpha\sin\beta$

$\dfrac{\partial y}{\partial\beta} = a\sinh\alpha\cos\beta$

$J = \begin{vmatrix} a\sinh\alpha\cos\beta & -a\cosh\alpha\sin\beta \\ a\sin\beta\cosh\alpha & a\sinh\alpha\cos\beta \end{vmatrix}$

$J = (a^3\sinh\alpha\cos\beta)^2 + (a\cosh\alpha\sin\beta)^2$

$J = a^2(\sin^2h\alpha\cos^2\beta + \cos^2h\alpha\sin^2\beta)$

$J = a^2(\sin^2h\alpha(1-\sin^2\beta) + (1+\sinh^2\alpha)\sin^2\beta)$

$= a^2(\sinh^2\alpha + \sin^2\beta)$ ans.

→ Properties of Jacobian

$$J J' = 1$$

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$$

$$\therefore \quad J' = 1/J$$

Q. If $x = r\cos\theta$
$y = r\sin\theta$
Show that $JJ' = 1$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)}$$

$\therefore$ $\frac{\partial x}{\partial r} = \cos\theta$ $\qquad$ $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad$ $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\therefore \quad J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Now

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and $r = \sqrt{x^2+y^2}$

$$\therefore \quad J' = \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2-y^2}}{\partial x} = -\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{+\cancel{2}y}{\cancel{2}\sqrt{x^2-y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{1}{\cancel{x^2}} = \frac{x}{x^2+y^2}$$

Now:

$$J' = \begin{vmatrix} \frac{-x}{\sqrt{x^2-y^2}} & \frac{+y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$J' = \frac{1}{r}$$

$$\Rightarrow \quad JJ' = r\left(\frac{1}{r}\right) = 1 \quad \text{ans}$$

→ Implicit Form of Jacobian

If $F_1(u, v, w, x, y, z) = 0$

$F_2(u, v, w, x, y, z) = 0$

$F_3(u, v, w, x, y, z) = 0$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}}$$

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$$J = \frac{\partial(v_1, v_2, v_3, \ldots v_n)}{\partial(x_1, x_2, x_3, \ldots x_n)}$$

$$J = (-1)^n \times \frac{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(x_1, x_2, \ldots x_n)}}{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(v_1, v_2, \ldots v_n)}}$$

Q. If

$$x + y + z = u$$
$$y + z = uv$$
$$z = uvw$$

then find J.

$$J = \frac{\partial(u_1, u_2, u_3, \ldots u_n)}{\partial(x_1, x_2, x_3, \ldots x_n)}$$

$$J = (-1)^n \times \frac{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(x_1, x_2, \ldots x_n)}}{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(u_1, u_2, \ldots u_n)}}$$

Q. If

$$x + y + z = u$$
$$y + z = uv$$
$$z = uvw$$

then find J.

$$F_1(x, y, z, u) = x + y + z - u$$
$$F_2(y, z, u, v) = y + z - uv$$
$$F_3(z, u, v, w) = z - uvw$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$\therefore \quad J = (-1)^3 \frac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}$$

$$\therefore \quad \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_3}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix}$$

$\therefore \quad \frac{\partial F_1}{\partial u} = -1 \qquad \frac{\partial F_2}{\partial u} = -v \qquad \frac{\partial F_3}{\partial u} = -vw$

$\frac{\partial F_1}{\partial v} = 0 \qquad \frac{\partial F_2}{\partial v} = -u \qquad \frac{\partial F_3}{\partial v} = -uw$

$\frac{\partial F_1}{\partial w} = 0 \qquad \frac{\partial F_2}{\partial w} = 0 \qquad \frac{\partial F_3}{\partial w} = -uv$

$$\Rightarrow \quad J = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$

$$= -1\,(+u^2v + (uw \times 0))$$

$$= -u^2v \quad \text{ans}$$

and

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

$\frac{\partial F_1}{\partial x} = 1 \qquad \frac{\partial F_2}{\partial x} = 0 \qquad \frac{\partial F_3}{\partial x} = 0$

$\frac{\partial F_1}{\partial y} = 1 \qquad \frac{\partial F_2}{\partial y} = 1 \qquad \frac{\partial F_3}{\partial y} = 0$

$\frac{\partial F_1}{\partial z} = 1 \qquad \frac{\partial F_2}{\partial z} = 1 \qquad \frac{\partial F_3}{\partial z} = 1$

$$\therefore \quad \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= +1(1) - 1(0) + 1(0) = 1$$

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$\therefore \quad J = (-1)^3 \dfrac{(-u^2v)}{1} = u^2v$ ans

Q. Show that

$JJ' = 1$ for

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$$

$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \dfrac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \dfrac{\partial z}{\partial r} = \cos\theta$

$\dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \dfrac{\partial z}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \dfrac{\partial z}{\partial \phi} = 0$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$J = \sin\theta\cos\phi(+r^2\sin^2\theta\cos\phi) -$$
$$r\cos\theta\cos\phi(-r\sin\theta\cos\theta\cos\phi) +$$
$$(-r\sin\theta\sin\phi)(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$J = r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi +$$
$$(-r\sin\theta\sin\phi)(-r\sin\phi)$$

$$J = r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$$
$$J = r^2\sin\theta\cos^2\phi(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$
$$J = r^2\sin\theta.$$

$$\therefore \quad J' = \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$$

Squaring and adding x, y and z

$$\Rightarrow \quad r^2 = x^2+y^2+z^2$$
$$r = \sqrt{x^2+y^2+z^2}.$$

→ Functionally dependent and functionally independent.

If $J = 0 \Rightarrow$ Functionally dependent
$J \neq 0 \Rightarrow$ Functionally Independent

Determinant → Polynomial → Equation → Roots

$P(x) = 0$ (Equation)

$P(x) = ax^2 + bx + c \Rightarrow$ Polynomial.

$P(x) = ax^2 + bx + c = 0 \Rightarrow$ Equation independent.

Ex: Show that the function

$$u = x - y + z$$
$$v = x + y - z$$
$$w = x^2 + y^2 + z^2 - 2yz$$

is functionally independent

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\therefore = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2x & 2y+2z & 2z-2y \end{vmatrix}$$

$$J = 1(\cancel{2z} - \cancel{2y} + \cancel{2y} - \cancel{2z}) + 1(2z - \cancel{2y} + \cancel{2x}) + 1(\cancel{2y} + 2z - \cancel{2x})$$

$$J = +4z$$

As; $J \neq 0$;

$\Rightarrow$ J is functionally independent.

→ Taylor's Theorem

Function of two variable :

If $f(x,y)$ is continuous and partially derivative at the point $x=a$, $y=b$. Then

$$f(x+h, y+k) = f(x,y) + (h\,\partial/\partial x + k\,\partial/\partial y) f(x,y) + \frac{1}{2!} \times (h\,\partial/\partial x + k\,\partial/\partial y)^2 f(x,y) + \frac{1}{3!} \times (h\,\partial/\partial x + k\,\partial/\partial y)^3 f(x,y) + \ldots$$

$$\frac{1}{(n-1)!} (h\,\partial/\partial x + k\,\partial/\partial y)^{n-1} f(x,y) + R_n$$

R_n → Remainder term
→ error

→ Taylor's Theorem is used to :

Get the expansion of a real function.

function is undefined

$$f(x+h, y+k) = f(x,y) + (h\,\partial/\partial x + k\,\partial/\partial y) f(x,y) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + k^2\frac{\partial^2}{\partial y^2} + 2hk\frac{\partial}{\partial x\,\partial y}\right) f(x,y) +$$

$$\frac{1}{3!}\left(h^3\frac{\partial^3}{\partial x^3} + k^3\frac{\partial^3}{\partial y^3} + 3h^2\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y} + 3k^2\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\right) f(x,y) + \ldots R_n$$

R_n → Approximation

Q. By using Taylor's Theorem ;

Find the expansion of

$$f(x,y) = e^x\cos y \text{ at the point } (0,0).$$

$f(x,y) = e^x\cos y$

$\frac{\partial f}{\partial x} = e^x\cos y \qquad \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 1$

$\frac{\partial f}{\partial y} = -e^x\sin y \qquad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$

$\frac{\partial^2 f}{\partial x^2} = e^x\cos y \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 1$

$\frac{\partial^2 f}{\partial y^2} = -e^x\cos y \qquad \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = -1$

$\frac{\partial^2 f}{\partial y\,\partial x} = -e^x\sin y \qquad \left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = 0$

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$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial f}{\partial y} = e^{x}\cos y(-e^{x}\sin y) \qquad \left.\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial f}{\partial x} = -e^{x}\cos y \cdot e^{x}\cos y \qquad \left.\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial f}{\partial y}\right|_{(0,0)} = -1.$$

$$\therefore \quad f(x+h, y+K) = f(x,y) + \left[h\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + K^2\frac{\partial^2 f}{\partial y^2}\right.$$

$$\left. + 2hK\frac{\partial f}{\partial x \partial y}\right] + \frac{1}{3!}$$

$$\left[h^3\frac{\partial^3 f}{\partial x^3} + K^3\frac{\partial^3 f}{\partial y^3} + 3h^2\frac{\partial^2 f}{\partial x^2} K\frac{\partial f}{\partial y} + 3h\frac{\partial f}{\partial x} \cdot K^2\frac{\partial^2 f}{\partial y^2}\right] + \cdots$$

$$\therefore \quad f(x+h, y+K) = 1 + h + \frac{1}{2!}(h^2 - K^2) + \frac{1}{3!}(h^3 - 3hK^2) + \cdots$$

To calculate the value of h and K

$\Rightarrow \quad x - x_0 = h$ and

$y - y_0 = K$

$\Rightarrow \quad x - 0 = h$

$y - 0 = K$

$$\Rightarrow f(x+h, y+K) = 1 + x + \frac{(x^2 - y^2)}{2!} + \frac{(x^3 - 3xy^2)}{3!} + \cdots$$

→ Maxima and minima of function of two variables

1. The point (A, B) is called relative or local minima if
$f(a+h, b+K) \geq f(a,b), \ \forall \ h$ and K.

The value of the function $f(x,y)$ at point (a,b) is called relative (local) minima value.

2. The point (a,b) is called relative (local) maximum if
$f(a+h, b+K) \leq f(a,b)$, for all h and K.

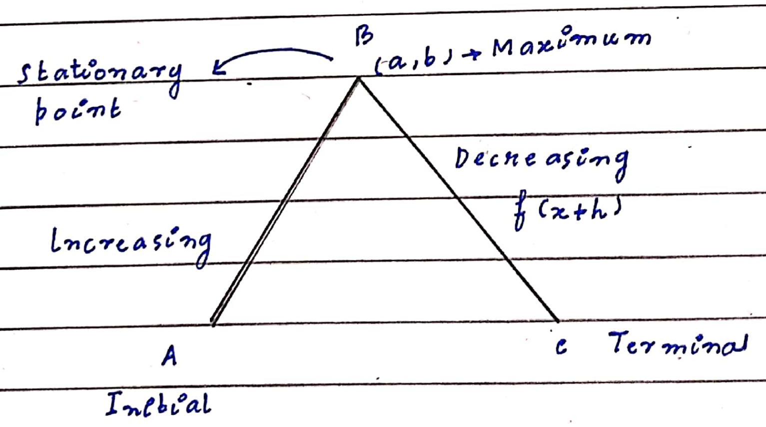
The value of the function $f(x,y)$ at the point (a,b) is called relative (local) maximum value.

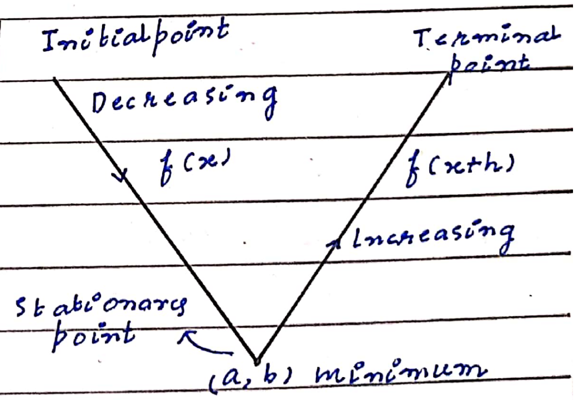
3. Stationary point : The point at which maximum / minimum occur is called point of the extreme or stationary point.

4. Saddle Point : The point at which the function is neither maxima or minima is called Saddle point.

5. Extreme Value : The value of the function at the point of extrem or stationary point is called extreme value.

□ $y = f(x)$

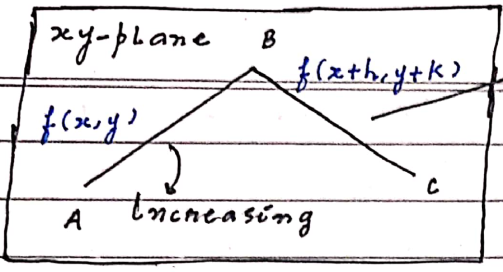




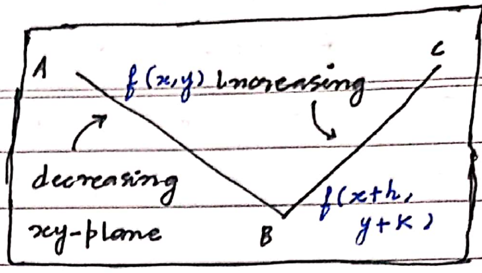
Distance : Terminal point - Initial point
: C - A
: Decreasing - Increasing = -ve
Local Maxima

Distance : C - A
: Increasing - decreasing
: +ve
: Local Minima

$z = f(x, y)$



→ Decreasing



$f(x+h, y+k) \leq f(x, y)$
Local Maxima

$f(x+h, y+k) \geq f(x, y)$
Local Minima

Saddle Point

Neither maxima nor minima

→ Necessary condition for maxima and minima

1. $rt - s^2 > 0$
2. If $rt - s^2 > 0$ and $r < 0 \Rightarrow$ Maxima
3. If $rt - s^2 > 0$ and $r > 0 \Rightarrow$ Minima
4. If $rt - s^2 = 0 \Rightarrow$ Saddle Point

→ By the partial defferentiation notation:

$$p = \partial z/\partial x$$
$$q = \partial z/\partial y$$
$$r = \partial^2 z/\partial x^2$$
$$s = \partial^2 z/\partial y \partial x$$
$$t = \partial^2 z/\partial y^2$$

→ Critical Point:

$$p = \partial z/\partial x = 0$$
$$q = \partial z/\partial y = 0$$

Q. Calculate minima and maxima for

$f(x, y) = x^3y^2(1-x-y)$

$p = \frac{\partial f}{\partial x} = -x^3y^2 + 3(1-x-y)x^2y^2$

$q = \frac{\partial f}{\partial y} = -x^3y^2 + 2(1-x-y)x^3y$

$r = \frac{\partial^2 f}{\partial x^2} = -3x^2y^2 - 3x^2y^2 + 6(1-x-y)xy^2$

$s = \frac{\partial^2 f}{\partial y \partial x} = -3x^2y^2 + 6(1-x-y)x^2y - 2x^3y$

$t = \frac{\partial^2 f}{\partial y^2} = -2x^3y - 2x^3y + 2(1-x-y)x^3$

$\Rightarrow$ For critical points

$p = 0$ and $q = 0$

$-x^3y^2 + 3(1-x-y)x^2y^2 = 0$ and $-x^3y^2 + 2(1-x-y)x^3y = 0$

on solving we get ;

$x = 0, \; y = 0$ and

$x = 1/2, \; y = 1/3$

At $(0,0)$

$r = 0$

$s = 0$

$t = 0$

$\Rightarrow \; rt - s^2 = 0 \Rightarrow$ Saddle point

At $(1/2, 1/3)$

$r = -6\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) + 6\left(1-\frac{1}{2}-\frac{1}{3}\right)\frac{1}{2}\left(\frac{1}{9}\right) = -\frac{1}{9}$

$t = -4\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) + 2\left(1-\frac{1}{2}-\frac{1}{3}\right)\left(\frac{1}{8}\right) = -\frac{1}{8}$

$s = -3\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) + 6\left(1-\frac{1}{2}-\frac{1}{3}\right)\frac{1}{4}\times\frac{1}{3} - 2\times\frac{1}{8}\times\frac{1}{3} = \frac{-1}{12}$

$rt - s^2$

$= \frac{1}{9} \times \frac{1}{8} - \frac{1}{144} = \frac{1}{144}$

As; $rt - s^2 > 0$ and $r < 0$

$\therefore$ There exists a maxima at $(1/2, 1/3)$ ans.

Q. Find maxima and minima

$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

$p = 4x - 4x^3$

$q = 4y^3 - 4y$

$r = 4 - 12x^2$

$s = 0$

$t = 12y^2 - 4$

To find stationary points

$p = 0$ and $q = 0$

$x = 0$ and $x = \pm 1$ $\quad$ $y = 0$ and $y = \pm 1$

$\therefore$ Stationary points are;

$(0,0)$, $(0, \pm 1)$

$(\pm 1, 0)$, $(\pm 1, \pm 1)$

$\therefore$ At $(0,0)$

$r = 4$

$t = -4$

$s = 0$

As, $rt - s^2 < 0$

$\Rightarrow$ $(0,0)$ acts as saddle point.

At $(0, \pm 1)$

$r = 4$ $\quad$ As, $rt - s^2 = 64 > 0$

$t = 8$ $\quad$ and $r > 0$

$s = 0$

$\Rightarrow$ $(0, \pm 1)$ acts as minima for $f(x,y)$

At $(\pm 1, 0)$;

$r = -8$

$t = -4$

$s = 0$

As $rt - s^2 = 32 > 0$

and $r < 0$

$\Rightarrow$ $(\pm 1, 0)$ acts as maxima for given function.

At $(\pm 1, \pm 1)$

$r = -8$

$t = 8$

$s = 0$

As, $rt - s^2 = -64 < 0$

$\Rightarrow$ $(\pm 1, \pm 1)$ is saddle point for given function.

Q. Find extreme value of the function;

$$f(x, y) = \sin x + \sin y + \sin(x+y)$$

$\therefore$ $p = \cos x + \cos(x+y)$

$q = \cos y + \cos(x+y)$

$r = -\sin x - \sin(x+y)$

$s = -\sin(x+y)$

$t = -\sin y - \sin(x+y)$

For stationary point:

$p = 0$ and $q = 0$

$\cos x + \cos(x+y) = 0$ and $\cos y + \cos(x+y) = 0$

$\Rightarrow$ on solving we get

$\cos x = \cos y$

$x = y$

$\Rightarrow$ $\cos x + \cos 2x = 0$

$\cos x + 2\cos^2 x - 1 = 0$

$\Rightarrow$ $x = \pi$ and $x = \pi/6$

$\Rightarrow$ Stationary points:

$(\pi/3, \pi/3)$ and (π, π)

$\therefore$ At $(\pi/3, \pi/3)$

$r = -\sqrt{3}$

$s = \sqrt{3}/2$

$t = -\sqrt{3}$

$\Rightarrow$ $rt - s^2$

$3 - \frac{3}{4} = \frac{9}{4} > 0$

$\therefore$ As $rt - s^2 > 0$ and $r < 0$

$\Rightarrow$ Point $(\pi/3, \pi/3)$ acts as maxima for given function

Extrem value of $f(x,y)$ at $(\pi/3, \pi/3)$:

$f(\pi/3, \pi/3) = 2\sin\pi/3 + \sin\left(\frac{2\pi}{3}\right)$

$= \sqrt{3} + \frac{\sqrt{3}}{2}$ ans

At (π, π)

$r = 0$

$s = 0$

$t = 0$

$\Rightarrow$ $rt - s^2 = 0$ (Langrange's Method)

$\therefore$

Q. Calculate the extreme value of the function

$f(x,y) = \sin x \sin y \sin(x+y)$

$p = \frac{\partial f}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y)$

$q = \frac{\partial f}{\partial y} = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y)$

$r = \frac{\partial^2 f}{\partial x^2 \partial y} = \cancel{-\sin x \sin y \sin(x+y)} - \cancel{\sin x \sin y \sin(x+y)}$
$\cos x \cos y \sin(x+y) + \sin x \cos y \cos(x+y) +$
$\cos x \sin y \cos(x+y) - \sin x \sin y \sin(x+y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y \sin(x+y) + \sin x \cos y \cos(x+y) +$
$\cos x \sin y \cos(x+y) - \sin x \sin y \sin(x+y)$

$t = -\sin x \sin y \sin(x+y) + \sin x \cos y \cos(x+y) +$
$\sin x \cos y \cos(x+y) - \sin x \sin y \sin(x+y).$

For stationary points;

$p=0$ and $q=0$;

$\cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) = 0$

$\cos x \sin(x+y) + \sin x \cos(x+y) = 0.$ — 1 $\Rightarrow \sin(2x+y) = 0$

or $\sin y = 0.$

$\sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y) = 0$

~~$\sin x \sin(x+y)$ +~~

$\cos y \sin(x+y) + \sin y \cos(x+y) = 0$ — 2 $\Rightarrow \sin(x+2y) = 0$

or $\sin x = 0.$

Q. Calculate the extreme value of
$f(x,y) = x^3 + y^3 - 3axy$

$\therefore \quad p = \frac{\partial f}{\partial x} = 3x^2 - 3ay$

$q = 3y^2 - 3ax$

$r = 6x$

$s = -3a$

$t = 6y$

For Critical points:

$p = 0$ and $q = 0$

$3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$

$x^2 - ay = 0$ $\qquad y^2 - ax = 0$

$x^2 = ay$ $\quad y = \frac{x^2}{a}$ $\qquad y^2 = ax$

x^4

$\cancel{\left(\frac{x}{y}\right)^2 = \left(\frac{y}{x}\right)}$

$\cancel{\left(\frac{x}{y}\right)^3 = 0}$

$y = \frac{x^2}{a}$ $\qquad \therefore \quad \frac{x^4}{a^2} = ax$

$x^3 = a^3$

$\therefore \quad a^2 = ay$ $\qquad x = a$

$y = a$

$\therefore$ Stationary point (a,a)

$\Rightarrow$ At (a,a):

$r = 6a$

$s = -3a$

$t = 6a$

$\Rightarrow \quad rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

As $rt - s^2 > 0$ and $r > 0$

$\Rightarrow$ (a,a) acts as minima for $f(x,y)$.

Extrum value $= a^3 + a^3 - 3a^3 = -a^3$ ans

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Langrange's Multipliers Method.

Let $f(x,y,z)$ with variable x, y, z where these variables are connected to $\phi(x,y,z)$ which is represent subjective to condition for the given function $f(x,y,z)$.

For the stationary point of the function $f(x,y,z)$

$$\frac{\partial f}{\partial x} = 0 \; ; \; \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} = 0.$$

By using total differentiation for the function $f(x,y,z)$.

$$\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} dz$$

and using total differentiation for the function $\phi(x,y,z)$

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$F(x,y,z) = \left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right] + \lambda \left[\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right] = 0$$

$\lambda \rightarrow$ Langrange's multiplier

$$\frac{\partial f}{\partial x} dx + \lambda \frac{\partial \phi}{\partial x} dx = 0$$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \; ;$$ Similarly ;

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$ and

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} dz = 0$$

To find the value of x, y, z and λ.
Calculate eq (1), (2) and (3).

Step 1: $F = f(x,y,z) + \lambda \phi(x,y,z)$

Step 2: $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$

Step 3: $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$; $\frac{\partial F}{\partial z} = 0$

Step 4: Calculate the value of x, y, z from 1, 2 and 3.

Step 5: Subsituting the value af x, y and z in subjected to condition $\phi(x,y,z)$.

Step 6: We will get value of λ.

Step 7: Subsitute the value of λ in step 5.

Step 8: We will get value of x, y and z.

Step 9: Subsitute the value of x, y and z in $f(x,y,z)$.

Example: Find the point upon the plane $ax+by+cz=p$ at which the function $x^2+y^2+z^2$ has a minimum. Find this minimum function f.

Given:

$$f = x^2+y^2+z^2 \quad \text{— (1)}$$
$$\phi = ax+by+cz=p \quad \text{— (2)}$$

Suppose:

$$F = f(x,y,z) + \lambda \phi(x,y,z).$$
$$F = (x^2+y^2+z^2) + \lambda(ax+by+cz-p)$$

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$\therefore \dfrac{\partial F}{\partial x} = 2x + \lambda a$ — (5)

$\dfrac{\partial F}{\partial y} = 2y + \lambda b$ — (6)

$\dfrac{\partial F}{\partial z} = 2z + \lambda c$ — (7)

For stationary point:

$\therefore \dfrac{\partial F}{\partial x} = 0$; $\dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$

$\Rightarrow x = \dfrac{-\lambda a}{2}$, $y = \dfrac{-\lambda b}{2}$ and $z = \dfrac{-\lambda c}{2}$

Subsituting the values of x, y and z in (2)

$\Rightarrow a\left(-\dfrac{\lambda a}{2}\right) + b\left(-\dfrac{\lambda b}{2}\right) + c\left(-\dfrac{\lambda c}{2}\right) = p$

$\therefore \lambda = \dfrac{-2p}{a^2+b^2+c^2}$

$\Rightarrow x = \dfrac{\cancel{2} ap}{(a^2+b^2+c^2)\cancel{2}}$; $y = \dfrac{bp}{a^2+b^2+c}$ $z = \dfrac{cp}{a^2+b^2+c^2}$

Putting x, y and z in $F(x, y)$

$f = \dfrac{p}{(a^2+b^2+c^2)}$. ans

Q. Find the maximum value of $f(x, y, z)$ $x^p y^q z^r$ subjected to condition $ax + by + cz = p + q + r$

Given:

$f = x^p y^q z^r$

$\phi = ax + by + cz = p + q + r$

$F = x^p y^q z^r + \lambda(ax + by + cz - p - q - r)$

$\therefore f = x^p y^q z^r$

$\therefore \ln f = p \ln x + q \ln y + r \ln z$

$$\therefore \quad \frac{\partial f}{\partial x} = p\frac{f}{x}$$

$$\frac{\partial f}{\partial y} = q\frac{f}{y}$$

$$\frac{\partial f}{\partial z} = r\frac{f}{z}$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} = p\frac{f}{x} + \lambda a$$

$$\frac{\partial F}{\partial y} = q\frac{f}{y} + \lambda b$$

$$\frac{\partial F}{\partial z} = r\frac{f}{z} + \lambda c$$

$\Rightarrow$ For stationary point

$$\therefore \quad x = \frac{-pf}{\lambda a}$$

$$y = \frac{-qf}{\lambda b}$$

$$z = \frac{-rf}{\lambda c}$$

$$\therefore \quad \phi = \frac{-pf}{\lambda} - \frac{qf}{\lambda} - \frac{rf}{\lambda} = p+q+r$$

$$\phi \Rightarrow \quad \lambda = -f$$

Now ; $\quad x = \frac{p}{a}$

$$y = q/b$$

$$z = r/c$$

$$\Rightarrow \quad f(x,y) = \left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r \text{ ans}$$

Q. If the temperature T at any point $T(x,y,z)$ in the space is

$$T(x,y,z) = Kxyz^2$$

Find the highest temperature on the surface sphere $x^2+y^2+z^2 = a^2$.

Given :

$$f(x,y,z) = Kxyz^2 \quad -1$$
$$\phi(x,y,z) = x^2+y^2+z^2 = a^2 \quad -2$$

$$\therefore \quad F = f(x,y,z) + \lambda\,\phi(x,y,z)$$
$$F = Kxyz^2 + \lambda(x^2+y^2+z^2-a^2) \quad -3$$

$$\therefore \frac{\partial F}{\partial x} = Kyz^2 + 2x\lambda \quad -4$$

$$\frac{\partial F}{\partial y} = Kxz^2 + 2y\lambda \quad -5$$

$$\frac{\partial F}{\partial z} = 2Kxyz + 2z\lambda \quad -6$$

For stationary points:

$$Kyz^2 + 2x\lambda = 0 \quad -7$$
$$Kxz^2 + 2y\lambda = 0 \quad -8$$
$$2Kxyz + 2z\lambda = 0 \quad -9$$

$$\therefore \quad Kxyz^2 + 2x^2\lambda = 0$$
$$Kxyz + 2y^2\lambda = 0$$
$$2Kxyz + 2z^2\lambda = 0$$

$$f + 2x^2\lambda = 0 \quad -10$$
$$f + 2y^2\lambda = 0 \quad -11$$
$$2f + 2z^2\lambda = 0 \quad -12$$

Adding 10, 11 and 12

$\Rightarrow \quad 4f + 2\lambda a^2 = 0$

$$\lambda = \frac{-4f}{2a^2} = \frac{-2f}{a^2}$$

Putting

$\therefore \quad \lambda = \frac{-2f}{a^2}$ in 7, 8 and 9

$$Kyz^2 + 2\left(\frac{-2f}{a^2}\right)x = 0$$

$$Kyz^2 = \frac{4fx}{a^2}$$

$$\cancel{K}\cancel{y}\cancel{z^2} = \frac{4 \times \cancel{K}x\cancel{y}\cancel{z^2} \times x}{a^2}$$

$$x = \pm \frac{a}{2}$$

Similarly :

$$y = \frac{a}{2}$$

and $\quad z = \frac{a}{\sqrt{2}}$

$\therefore$ Putting x, y and z in $f(x, y, z)$

$$f = xyz^2 = \frac{Ka^4}{8} \quad \text{ans.}$$

Q. Find maximum and minimum distance of point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

$$f(x, y, z) = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

and $\quad \phi(x, y, z) = x^2 + y^2 + z^2 = 1$

$\therefore \quad F = f(x, y, z) + \lambda \phi(x, y, z)$

$$F = \left(\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2} + \lambda \cdot (x^2 + y^2 + z^2 - 1)\right)$$

$$\frac{\partial F}{\partial x} = \frac{1 \times \cancel{2}(x-3)}{\cancel{2}\sqrt{(x-3)^2 + \cdots}} + 2\lambda x$$

and so on.

$$\Rightarrow \quad \frac{\partial F}{\partial x} = 0$$

$$\therefore \quad \frac{1(x-3)}{f} + 2\lambda x = 0$$

$$x = \frac{3}{2\lambda f + 1} \quad \text{and}$$

$$y = \frac{4}{2\lambda f + 1} \quad \text{and}$$

$$z = \frac{12}{2\lambda f + 1}$$

and

$\lambda = 6/f$ and $\lambda = -7/f$.

↑ Maximum distance ↑ Minimum distance