Sequence and Series

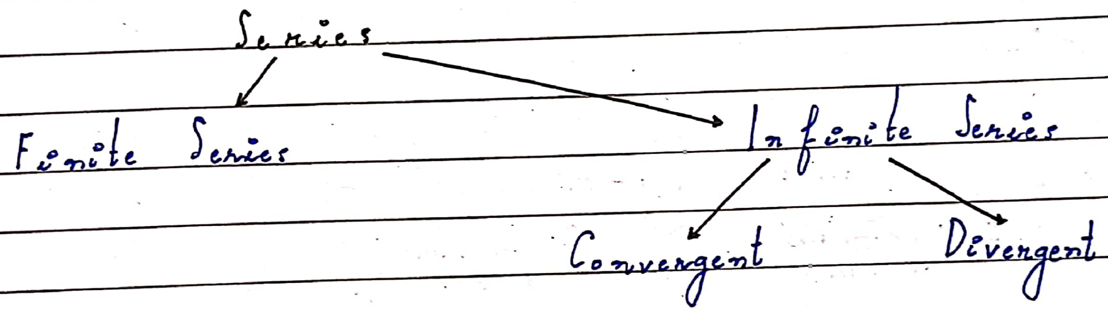
→ **Series :** An expression :

$x_1 + x_2 + \dots u_n + \dots$, where

every term followed by another term under some definite rule, then this expression is known as series.

→ How will you construct series ?

With the help of the sequence we can form series.



→ **Convergent Series :**

→ N^{th} partial sum

$$S = u_1 + u_2 + u_3 + \dots + u_n + \dots$$

Take n^{th} term of the infinite series

$$S_n = u_1 + u_2 + u_3 + \dots + u_n.$$

This S_n is known as n^{th} partial sum.

$$\lim_{n \to \infty} S_n = S$$

→ **Convergent Series**

A series $\sum u_n = S$ is said to be convergent series if first n^{th} term of the infinite series tends to definite finite unique Limit $S_n \to S$ as n tends to infinity.

$$\lim_{n \to \infty} S_n = S \left[\begin{matrix}\text{definite,} \\ \text{finite,} \\ \text{unique}\end{matrix}\right]$$

→ Divergent Series

A series $\sum u_n = S$ is said to be divergent if the first n^{th} term S_n of the infinite series tends to $+\infty$ or $-\infty$ as n tends to infinite.

$$\lim_{n \to \infty} S_n = S \quad (+\infty \text{ or } -\infty)$$

→ Oscillatory Series

A series $\sum U_n = S$ is said to be oscillatory series if the first n^{th} term of the infinite series S neither tends to finite nor $+\infty$ or $-\infty$.

$$\lim_{n\to\infty} S_n = \begin{cases} \text{neither finite} \\ \text{nor } +\infty \text{ or } -\infty \end{cases}$$

→ Geometric Series

$$S = 1 + r + r^2 + r^3 + \dots + r^n + \dots$$

Case 1 : If $r > 1$

$$S_n = \frac{r^n - 1}{r - 1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n - 1}{r - 1} = \infty \quad \text{(Divergent)}$$

Case 2 : If $r < 1$

$$S_n = \frac{1 - r^n}{1 - r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} = (1 - r)^{-1} \quad \text{(Convergent)}$$

$|r| < 1$.

Scanned with OKEN Scanner

Case 3: If $r = 1$, then

$$G_s = 1 + r^2 + r^3 + r^4 + \dots + r^n + \dots$$

$r = 1$

$$G_s = 1 + 1 + 1 + 1 + \dots + 1 + \dots$$

This series is known as constant series and constant series always divergent.

Case 4: If $r = -1$, then

$$G_s = 1 + (-1) + (-1)^2 + (-1)^3 + \dots + (-1)^n + \dots$$

$$\therefore \quad S_n = \frac{1 - r^n}{1-(-1)} = \frac{1-(-1)^n}{2}$$

Alternating Series

$\therefore$ If $n = \text{odd}$

$S_n = 0$

and

If $n = \text{even}$

$S_n = 1$

As;

$$\lim_{n \to \infty} S_n = \begin{cases} 0, & n \to \text{odd} \\ 1, & n \to \text{even} \end{cases}$$

$\therefore$ This is an oscillatory series.

Case 5: If $r < -1$, then

$$S_n = \frac{1 - r^n}{1 - r}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = -\infty \text{ (Divergent)}$$

→ Comparison Test

If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are positive term series,

1. Convergence of series $\sum V_n$ implies convergence of the series $\sum U_n$.
2. Divergence of series $\sum U_n$ implies divergence of the series $\sum V_n$.

→ Auxiliary Series

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

1. Convergent if $p > 1$.
2. Divergent if $p \leq 1$.

Q. Test the convergence of the series

Example: $\frac{1}{a.1^2+b} + \frac{2}{a.2^2+b} + \frac{3}{a.3^2+b} + \dots$

$\therefore$ n^{th} term $= \frac{n}{a.n^2+b}$

$\therefore \sum_{n=1}^{\infty} U_n = S_n = \sum_{n=1}^{\infty} \frac{n}{an^2+b}$

To calculate the auxillary series

$$V_n = \frac{\text{highest power of } n \text{ in the numerator of } U_n}{\text{highest power of } n \text{ in the denominator of } U_n}$$

$$V_n = \frac{n}{n^2} = \frac{1}{n}$$

$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n}$ $\therefore p = 1$.

Divergent Series

$$\frac{U_n}{V_n} = \frac{\frac{n}{an^2+b}}{1/n}$$

$$\lim_{n \to \infty} \frac{U_n}{V_n} = \lim_{n \to \infty} \frac{n^2}{an^2+b} = \lim_{n \to \infty} \frac{1}{a+\frac{b}{n^2}} = \frac{1}{a}$$

$\therefore$ If $\frac{1}{a} > 1 \Rightarrow$ Divergent

$\frac{1}{a} < 1 \Rightarrow$ Convergent

But, we know that from the comparison $\sum U_n$ and $\sum V_n$ converges and diverges together.

Here, $\sum V_n = \sum \frac{1}{n}$ is divergent. Hence, the given series is also divergent.

Sequence and Series

→ **Series:** An expression:

$x_1 + x_2 + \dots u_n + \dots$, where every term followed by another term under some definite rule, then this expression is known as series.

→ How will you construct series?

With the help of the sequence we can form series.

- Series
  - Finite Series
  - Infinite Series
    - Convergent
    - Divergent

→ **Convergent Series:**

→ N^{th} partial sum

$S = u_1 + u_2 + u_3 + \dots + u_n + \dots$

Take n^{th} term of the infinite series

$S_n = u_1 + u_2 + u_3 + \dots + u_n.$

This S_n is known as n^{th} partial sum.

$$\lim_{n \to \infty} S_n = S$$

→ **Convergent Series**

A series $\sum u_n = S$ is said to be convergent series if first n^{th} term of the infinite series tends to definite finite unique limit $S_n \to S$ as n tends to infinity.

$$\lim_{n \to \infty} S_n = S \left[\begin{array}{l}\text{definite,} \\ \text{finite,} \\ \text{unique}\end{array}\right]$$

→ Divergent Series

A series $\sum u_n = S$ is said to be divergent if the first n^{th} term S_n of the infinite series tends to $+\infty$ or $-\infty$ as n tends to infinite.

$$\lim_{n \to \infty} S_n = S \quad (+\infty \text{ or } -\infty)$$

→ Oscillatory Series

A series $\sum U_n = S$ is said to be oscillatory series if the first n^{th} term of the infinite series S neither tends to finite nor $+\infty$ or $-\infty$.

$$\lim_{n \to \infty} S_n = \begin{cases} \text{neither finite} \\ \text{nor } +\infty \text{ or } -\infty \end{cases}$$

→ Geometric Series

$$S = 1 + r + r^2 + r^3 + \dots + r^n + \dots$$

Case 1 : If $r > 1$

$$S_n = \frac{r^n - 1}{r - 1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{r^n - 1}{r - 1} = \infty \quad \text{(Divergent)}$$

Case 2 : If $r < 1$

$$S_n = \frac{1 - r^n}{1 - r}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} = (1 - r)^{-1} \quad \text{(Convergent)}$$

$|r| < 1$.

Case 3: If $r = 1$, then

$$G_s = 1 + r^2 + r^3 + r^4 + \dots + r^n + \dots$$

$r = 1$

$$G_s = 1 + 1 + 1 + 1 + \dots + 1 + \dots$$

This series is known as constant series and constant series always divergent.

Case 4: If $r = -1$, then

$$G_s = 1 + (-1) + (-1)^2 + (-1)^3 + \dots + (-1)^n + \dots$$

$$\therefore \quad S_n = \frac{1 - r^n}{1 - (-1)} = \frac{1 - (-1)^n}{2}$$

Alternating Series

$\therefore$ If $n = \text{odd}$

$S_n = 0$

and

If $n = \text{even}$

$S_n = 1$

As:

$$\lim_{n \to \infty} S_n = \begin{cases} 0, & n \to \text{odd} \\ 1, & n \to \text{even} \end{cases}$$

$\therefore$ This is an oscillatory series.

Case 5: If $r < -1$, then

$$S_n = \frac{1 - r^n}{1 - r}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = -\infty \text{ (Divergent)}$$

→ Comparison Test

If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are positive term series,

1. Convergence of series $\sum V_n$ implies convergence of the series $\sum U_n$.
2. Divergence of series $\sum U_n$ implies divergence of the series $\sum V_n$.

→ Auxiliary Series

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

1. Convergent if $p > 1$.
2. Divergent if $p \leq 1$.

Q. Test the convergence of the series

Example: $\frac{1}{a.1^2+b} + \frac{2}{a.2^2+b} + \frac{3}{a.3^2+b} + \dots$

$\therefore$ n^{th} term $= \frac{n}{a.n^2+b}$

$$\therefore \sum_{n=1}^{\infty} U_n = S_n = \sum_{n=1}^{\infty} \frac{n}{an^2+b}$$

To calculate the auxiliary series

$$V_n = \frac{\text{highest power of } n \text{ in the numerator of } U_n}{\text{highest power of } n \text{ in the denominator of } U_n}$$

$$V_n = \frac{n}{n^2} = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \therefore p = 1.$$

Divergent Series

$$\frac{U_n}{V_n} = \frac{\frac{n}{an^2+b}}{1/n}$$

$$\lim_{n \to \infty} \frac{U_n}{V_n} = \lim_{n \to \infty} \frac{n^2}{an^2+b} = \lim_{n\to\infty} \frac{1}{a+\frac{b}{n^2}} = \frac{1}{a}$$

$\therefore$ If $\frac{1}{a} > 1 \Rightarrow$ Divergent

$\frac{1}{a} < 1 \Rightarrow$ Convergent

But, we know that from the comparison $\sum U_n$ and $\sum V_n$ converges and diverges together.

Here, $\sum V_n = \sum \frac{1}{n}$ is divergent. Hence, the given series is also divergent.

PAGE NO.
DATE: / /

$\therefore$ The given series is divergent.

Q ⇒ Test the convergence of the series

$$\Sigma \frac{n^2}{\sqrt{n}-1}$$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} U_n = \sum_{\lim n\to\infty}^{\infty} \frac{n^2}{\sqrt{n}-1}$$

To calculate auxiliary series,

$$V_n = \frac{n^2}{n^{1/2}} = n^{2-1/2}$$

$$V_n = n^{3/2}$$

$$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{3/2} = \frac{1}{n^{-3/2}}$$

Now compare with auxiliary series

By p-test

If $p \le 1$ (divergent)

$$p = -3/2 < 1$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{\frac{n^2}{\sqrt{n}-1}}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}-1} = 1$$

19/09/24

PAGE NO.
DATE: / /

Cauchy n^{th} ~~rule~~ root test (जब n की power dekhna)

If ΣU_n be the terms of the positive series, and

$$\lim_{n\to\infty} U_n^{1/n} = l, \text{ then}$$

ΣU_n is convergent if $l<1$
ΣU_n is divergent if $l>1$
For $l=1$, test fail.

Ex-1) Test the series $\Sigma U_n = \Sigma \left(\frac{n+1}{n+2}\right)^n$

= Given,

$$U_n = \left(\frac{n+1}{n+2}\right)^n$$

$$(U_n)^{1/n} = \left(\frac{n+1}{n+2}\right)^{n \cdot \frac{1}{n}}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{2}{n}\right)} = 1$$

(Test failed)

Ex-2) Test the series $\Sigma U_n = \Sigma \left(\frac{n+1}{n+3}\right)^{n^2}$

= Given, $U_n = \left(\frac{n+1}{n+3}\right)^{n^2}$

$$(U_n)^{1/n} = \left(\frac{n+1}{n+3}\right)^{n^2 \cdot \frac{1}{n}}$$

PAGE NO.
DATE: / /

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+3}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+3}\right)^{n}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{n}}$$

$$\boxed{\begin{aligned}&\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e\\ &\left[\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{\frac{n}{3}}\right]^{3} = e^3\end{aligned}}$$

$$= \frac{e}{e^3} = \frac{1}{e^2}$$

$$\because \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{x} = e^x$$

according to cauchy n^{th} ~~roote~~ test, n is <1, hence the given series is convergent.

Ex-3) Test the series $\sum u_n = \sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

= Given,

$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{1/n} = \left(\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}\right)^{1/n}$$

PAGE NO.
DATE: / /

$$= \lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)^{n}}\right) = \frac{1}{e}$$

$\therefore$ The given series is convergent

Q. Test the series $\sum u_n = \sum \frac{1}{(\log n)^n}$

$\Rightarrow$ Given, $u_n = \frac{1}{(\log n)^n}$

~~$u_n = (\log n)^{-n}$~~

$$(u_n)^{1/n} = \left(\frac{1}{\log n}\right)^{\cancel{n}/\cancel{n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{\log n}\right)$$

$$= \frac{1}{\infty} = 0 < 1$$

$\therefore$ The given series is convergent.

Ex-4) Test the series $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$

$$= \quad u_n = \left(\frac{(n+1)^{n+1}}{(n)^{n+1}} - \frac{(n+1)}{(n)}\right)^{-n}$$

$$(u_n)^{1/n} = \left(\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right)^{-1}$$

$$= \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right)^{-1}$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{n}\left(\left(\frac{n+1}{n}\right)^{n} - 1\right)\right]^{-1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-1} \left(\left(1+\frac{1}{n}\right)^{n} - 1\right)^{-1}$$

$$= 1\,(e-1)^{-1}$$

$$= \frac{1}{e-1} < 1$$

∴ Hence, the given series is convergent.

<u>Ratio Test</u> (D' Alembert Ratio Test)

whenever we get factorial ⇒ we deal it with ratio test.

If Σu_n is the positive term series and $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l$, then

1. Σu_n is convergent if $l<1$
2. Σu_n is divergent if $l>1$
3. If $l=1$ test fails.

or

If Σu_n is the positive terms series, and $\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = l$, then

1. Σu_n is convergent if $l>1$.
2. Σu_n is divergent if $l<1$.
3. Test fails if $l=1$

Ex- Test the series $1 + 3x + 5x^2 + 7x^3 + \dots$

⇒ $u_n = (2n-1)x^{n-1}$

$u_{n+1} = (2(n+1)-1)x^n$

$= (2n+1)x^n$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)x^n}{(2n-1)x^{n-1}} = \cancel{\left(\frac{2n+1}{2n+1-2}\right)x}$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{2n-1+2}{2n-1}\right)x$$

$$\frac{u_{n+1}}{u_n} = \left(1+\frac{2}{2n-1}\right)x$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{2}{2n-1}\right)\right||x|$$

$$= |(1)|\cdot|x| = |x|$$

$$|x| = \begin{cases} >1 \\ <1 \\ =1 \end{cases}$$

$|x|<1$ ⇒ so, that it

PAGE NO.
DATE: / /

$$= \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x|$$

$x < 1$ (Convergent)
$x > 1$ (Divergent)
$x = 1$ (Test fail)

$1 + 3x + 5x^2 + 7x^3 + \cdots$

for $x = 1$

The above series becomes

$1 + 3 + 5 + 7 + \cdots$

(Constant series)

Constant series is always divergent

$\therefore$ The given series is convergent for $x < 1$, divergent for $x \geq 1$.

Q. $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots \frac{x^n}{n(n+1)} + \cdots$

$$\Rightarrow u_n = \frac{x^n}{n(n+1)}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

By using ratio test, $\frac{u_{n+1}}{u_n}$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{\cancel{(n+1)}(n+2)} \cdot \frac{n\cancel{(n+1)}}{x^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{nx}{n+2}$$

PAGE NO.
DATE: / /

$$= \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{nx}{n+2}\right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{\cancel{n}}{\cancel{n}} \left(\frac{1}{1 + \frac{2}{n}} \right) \right| = |x| \cdot \lim_{n\to\infty} |1|$$

$$= x = |x|$$

$x < 1$ (Convergent)
$x > 1$ (Divergent)
$x = 1$ (Test fail)

For $x = 1$, the given series becomes

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

= By using comparison test

$$\Sigma u_n = \Sigma \frac{1}{n(n+1)}$$

$$u_n = \frac{1}{n(n+1)}$$

$$u_n = \frac{n^0}{n^2+n}$$

$$v_n = \frac{n^0}{n^2} = \frac{1}{n^2}$$

$\Sigma v_n = \Sigma \frac{1}{n^2}$ (Auxiliary series).

By using p-test

$\Sigma \frac{1}{n^p}$, if $p > 1$ (Convergent)
$p \leq 1$ (divergent)

$p = 2 > 1$ (Convergent)

PAGE NO.
DATE: / /

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{n+1}$$

$= 1$ (finite & unique)

By using comparison test

Σu_n and Σv_n converge & diverge together but here $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

I	II
$\Sigma \frac{1}{n^p}$ (Auxiliary) series.	$\lim_{n\to\infty} \frac{u_n}{v_n} = l$

→ If any form is divergent, series will be divergent.
→ If both are convergent, then only series will be convergent.

Hence, the given series is divergent.

Ex- Test the series: $\Sigma \frac{n+1}{n^3} x^n$.

⇒ If n^{th} term is not calculated by use $a_n = a + (n-1)d$ separately for both numerator & denominator
→ Many times, we do not get value while using $n=1, n=0, n=2$ etc.

PAGE NO.
DATE: / /

Given,

$$\Rightarrow u_n = \frac{n+1}{n^3} x^n$$

$$u_{n+1} = \frac{(n+2)}{(n+1)^3} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^3} x^{n+1} \cdot \frac{n^3}{(n+1) \cdot x^n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{n^3 (n+2) x}{(n+1)^4} = \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^4} \cdot x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left| \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^4} \right| |x|$$

$= |x|$

If $x > 1$ (divergent)
$x < 1$ (convergent)
$x = 1$ (Test fails)

For $x = 1$, the given series becomes

$$= \Sigma \frac{(n+1)}{n^3} \cdot x^n$$

By using comparison test, the auxiliary series becomes

$$= \Sigma u_n = \Sigma \frac{n+1}{n^3}.$$

PAGE NO.
DATE: / /

$V_n = \frac{n}{n^3} = \frac{1}{n^2}$

$\Sigma V_n = \Sigma \frac{1}{n^2}$ (Auxiliary series)

By using p-test

$\Sigma \frac{1}{n^p} = \Sigma \frac{1}{n^2}$, here

$p = 2 > 1$ (Convergent)

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$

Ex- Test the series $\Sigma \frac{(n+3)!}{(n+5)!} x^n$

Given $u_n = \frac{(n+3)!}{(n+5)!} \cdot x^n$

$u_{n+1} = \frac{(n+4)!}{(n+6)!} x^{n+1}$

$\frac{u_{n+1}}{u_n} = \frac{(n+4)!}{(n+6)!} \cdot \frac{(n+5)!}{(n+3)!} \cdot x^1$

$\frac{u_{n+1}}{u_n} = \frac{n+4}{n+6} \cdot x^1 \Rightarrow \lim_{n\to\infty} \left(\frac{1+\frac{4}{n}}{1+\frac{6}{n}}\right) = 1$

$= \lim |x|$ $\Rightarrow$ $x > 1$ (Divergent)
$x < 1$ (Convergent)
$x = 1$ (Test fail)

PAGE NO.
DATE: / /

For $x = 1$

$= \cancel{\frac{4!}{5!}}\ \cancel{\Sigma u_n} = \Sigma \frac{(n+3)!}{(n+5)!}$

$\Sigma V_n = \Sigma \frac{1}{n^2}$ (Auxiliary series)

By using p-test
$p = 2 > 1$ (Convergent)

$\lim_{n\to\infty} \frac{u_n}{V_n} = 1$

Thus, the given series becomes divergent for $x = 1$.

$\therefore$ Given series $x > 1$ (Divergent)
$x < 1$ (Convergent)
$x = 1$ (Divergent)

Absolutely Convergent

A series Σu_n is said to be absolutely convergent if $\Sigma |u_n|$ is convergent.

Semi Convergent / Conditionally convergent

A series Σu_n is said to be semi-convergent if Σu_n is convergent, but $\Sigma |u_n|$ is divergent.

Lebnitiz Test

PAGE NO.
DATE: / /

An alternative ~~tes~~ series $\Sigma (-1)^{n+1} u_n$ is said to be convergent if

$$\lim_{n\to\infty} (-1)^{n+1} u_n = 0$$

Ex- Test the series:

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$$

$$\Rightarrow \Sigma (-1)^{n+1} u_n = \Sigma (-1)^{n+1} \frac{1}{2^{n-1}} = \Sigma (-1)^n \frac{1}{2^n}$$

By using Leibnitz Test,

$$\lim_{n\to\infty} (-1)^{n+1} u_n = \lim_{n\to\infty} (-1)^n \frac{1}{2^n}$$

$$= 0$$

Here, Σu_n is convergent

Step 2: $\Sigma |u_n| = \Sigma |(-1)^{n+1} u_n|$

$$= \Sigma |u_n| \cancel{+} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$> \Sigma u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= 1 + r + r^2 + \cdots$$

$$r = \frac{1}{2}$$

PAGE NO.
DATE: / /

$$S_n = \frac{1-r^n}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1-r^n}{1-r}\right) = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \doteq 2$$

The value is fixed & unique.
Hence, the series $\Sigma|U_n|$ is convergent.

$\Rightarrow$ Hence, absolutely convergent.

Q. Test the convergence $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

$-\frac{1}{6}+\dots$

Step 1 By using Leibnitz rule

$$\lim_{n\to\infty}(-1)^{n+1}U_n = \lim_{n\to\infty}(-1)^{n+1}\frac{1}{n}$$
$$=0$$

$\therefore \Sigma U_n$ is convergent

Step 2 $\Sigma|U_n| = \Sigma\left|(-1)^{n+1}\frac{1}{n}\right|$

$$\Sigma|U_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$$

$V_n = \frac{1}{n}$

Auxiliary series

$$\Sigma V_n = \Sigma\frac{1}{n^{p=1}}$$

PAGE NO.
DATE: / /

By p-test,
ΣV_n is divergent for $p\le 1$.
Hence, given series is semi-convergent.

Q. Test the series
$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\dots$$

Step 1: By using Leibnitz rule

$$\lim_{n\to\infty}(-1)^{n+1}U_n = \lim_{n\to\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$$
$$=0$$

$\therefore \Sigma U_n$ is convergent

Step 2: $\Sigma|U_n| = \Sigma\left|(-1)^{n+1}\frac{1}{\sqrt{n}}\right|$

$$\Sigma|U_n| = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots$$

$$V_n = \frac{1}{(n)^{1/2}}$$

Auxiliary series

$$\Sigma V_n = \Sigma\frac{1}{n^{p=1/2}}$$

By p-test $\Rightarrow p<1$

ΣV_n is divergent for $p\le 1$

Hence, the given series semi-convergent.