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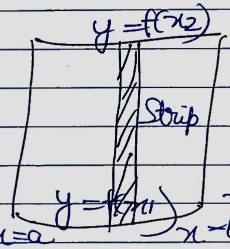
Multiple Integral

funcⁿ के पास जितने integral → utne independent variables

$$\int_{y=a,\ x=var.}^{y=b,\ x=var.}\int f(x,y)\,dx\,dy$$

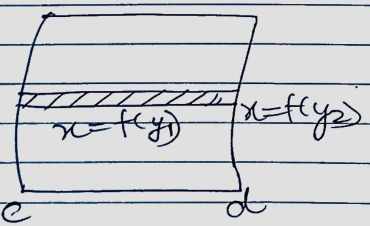
Ex. $$\int_{y=2}^{y=3}\int_{x=y}^{x=y^2} f(x,y)\,dx\,dy$$



$$\int_{x=a}^{x=b}\int_{y=f(x_1)}^{y=f(x_2)} f(x,y)\,dy\,dx$$

पहला variable, दूसरा constant

strip जिसके along जाएगा उसका integration पहले होगा।



$$= \int_{y=a}^{y=b}\int_{x=f(y_1)}^{x=f(y_2)}$$

n-dimension को हम single dimension में change करेंगे ⇒ because all formulas are defined for ~~signal~~ single dimension.

Beta (β) and Gamma (Γ) Functions

Gamma Function: Generalization of a factorial function. $\Gamma(n)$

Notation: $\boxed{\Gamma(n)}$

Formula: $\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt$, where $t>0$, $n>0$

$t \to$ parameter

Properties

① $\Gamma(n+1) = n\Gamma(n)$

② $\Gamma(n) = (n-1)\Gamma(n-1)$

③ $\Gamma(n+1) = n!$

④ $\Gamma(n)\,\Gamma(1-n) = \Gamma($ → Table is reqd

Q Find the value of $\Gamma(5)$

$= \Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(5) = (5-1)\,\Gamma(5-1)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

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Ex- $\Gamma\left(\frac{3}{2}\right) = ?$

$= \Gamma\left(\frac{3}{2}\right) = (n-1)\Gamma(n-1)$

$= \left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

Ex- $\Gamma\left(\frac{7}{2}\right) = ?$

$= \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

$= \frac{15}{8}\sqrt{\pi}$

Ex- $\Gamma\left(\frac{13}{2}\right) = ?$

$= \Gamma\left(\frac{13}{2}\right) = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$= \frac{10395}{64}\sqrt{\pi} = \frac{10395}{64}\sqrt{\pi}$

Ex- $\Gamma\left(-\frac{1}{2}\right) = ?$ Ex- $\Gamma\left(-\frac{3}{2}\right) = ?$

$= \Gamma\left(\frac{1}{2}\right) = (n-1)\Gamma(n-1)$

$= \left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right)$

$= -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

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$= -2\Gamma\left(\frac{1}{2}\right) = \Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Q- $\Gamma\left(-\frac{5}{2}\right) = ?$

$= \Gamma\left(-\frac{1}{2}\right) = \left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right)$

$= \Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)$

$= \Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}-1\right)\Gamma\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}\right)$

$= \Gamma\left(\frac{1}{2}\right) = \left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)$

$= \Gamma\left(\frac{1}{2}\right) = \left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right)$

$\Rightarrow \sqrt{\pi}\times\left(\frac{2}{5}\right)(-1) = -\frac{8}{15}\sqrt{\pi} \rightarrow$ Ans.

Ex- Prove $\Gamma(n+1) = n!$

By using Γ func^n formula

$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt$

$\Gamma(n+1) = \int_0^\infty e^{-t}\, t^{n}\, dt$

For $n=2$

$\Gamma(2+1) = \int_0^\infty e^{-t}\, t^2\, dt$

$-\infty\ -4\ -3\ -2\ -1\ 0\ 1\ 2\ 3\ 4\ \infty$

$$\Gamma(3) = \left[t^2 \cdot \int e^{-t}\right]_0^\infty - \int_0^\infty \left(\frac{dt^2}{dt} \int e^{-t}\, dt\right) dt$$

$$\Gamma(3) = \left[t^2 \cdot (-e^{-t})\right]_0^\infty - \int_0^\infty (2t)(-e^{-t})\, dt$$

$$\Gamma(3) = \left[t^2 \cdot (-e^{-t})\right]_0^\infty + 2\int_0^\infty t \cdot e^{-t}\, dt$$

$$\Gamma(3) = \lim_{t\to\infty} t^2 \cdot (e^{-t}) + 0 \times e^{-0} + 2\int_0^\infty t\, e^{-t}\, dt$$

$$\Gamma(3) = -\lim_{t\to\infty} t^2 \cdot \lim_{t\to\infty} (e^{-t}) + 0 + 2\int_0^\infty e^{-t}\, t\, dt$$

$$\Gamma(3) = \infty \times 0 + 0 + 2\int_0^\infty e^{-t}\, t\, dt$$

Again by using integration by parts

$$= 2\left[\left[t \cdot \int e^{-t}\right]_0^\infty - \int_0^\infty \left(\frac{dt}{dt} \int e^{-t}\right) dt\right]$$

$$= 2 \times 0 - 2\int_0^\infty e^{-t}\, dt$$

$$= -2\left[-e^{-t}\right]_0^\infty = +2\left[\lim_{t\to\infty}(-e^{-t}) + e^{-0}\right]$$

$$\Gamma(3) = 2[0+1] = 2!$$

Similarly, $\Gamma(n+1) = n!$

\# $\Gamma(n) = \int_0^\infty e^{-t} \cdot t^{n-1} \cdot dt$

$$\Gamma(n+1) = \int_0^\infty e^{-t} \cdot t^{n+1-1}\, dt$$

$$= \int_0^\infty e^{-t} \cdot t^n\, dt$$

$$= \left[t^n \cdot \int e^{-t} \cdot dt\right]_0^\infty - \int_0^\infty \left(\frac{dt^n}{dt} \int e^{-t} \cdot dt\right) dt$$

$$= \left[t^n \cdot \int e^{-t} \cdot dt\right]_0^\infty - \int_0^\infty n t^{n-1} (-e^{-t})\, dt$$

$$= 0 + n\int_0^\infty t^{n-1} \cdot e^{-t} \cdot dt$$

$$= n(n-1)\int_0^\infty t^{n-2} \cdot e^{-t} \cdot dt$$

$$= n(n-1)(n-2)\int_0^\infty t^{n-3} e^{-t} \cdot dt$$

$$= n(n-1)(n-2)(n-3) \cdots 2\int_0^\infty e^{-t}\, dt$$

Q. Evaluate $\int_0^\infty e^{-st}\, t^2 \cdot dt$

$$= \Gamma(n) = \int_0^\infty e^{-t} \cdot t^{n-1} \cdot dt$$

By using Gamma Func^n. formula

$$\Gamma(n) = \int_0^\infty e^{-st}\, t^2 \cdot dt$$

By using substitution method.

Let $st = u$

$s\,dt = du$

$dt = \frac{1}{s} \cdot du$

when $t = 0$

$s \times 0 = u \Rightarrow u = 0$

when $t = \infty$

$s \times \infty \Rightarrow u = \infty$

$$\Gamma(n) = \int_{u=0}^{u=\infty} e^{-u} \left(\frac{u}{s}\right)^2 \cdot \frac{1}{s} \cdot du$$

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$= \frac{1}{S3} \int_0^\infty e^{-u} \cdot u^2 \cdot du$

$= \frac{1}{S3} \int_0^\infty e^{-u} \cdot u^{3-1} \cdot du$

$\Gamma(n) = \int_0^\infty e^{-t} \cdot t^{n-1} dt$

$= \frac{1}{S3} \Gamma(3) \quad = \quad = \frac{\Gamma(n+1)}{S^{n+1}}$

Q. Evaluate $\int_0^\infty e^{-t^2} \cdot t^5 \cdot dt$

By using substitution method

$t^2 = u \quad \Rightarrow t = \sqrt{u}$

$2t\,dt = du$

$dt = \frac{du}{2t} = \frac{du}{2\sqrt{u}}$

= When $t=0, u=0$
$t=\infty, u=\infty$

= The given integral becomes

$\int_0^\infty e^{-u} \cdot u^{5/2} \cdot \frac{du}{2\sqrt{u}}$

$= \frac{1}{2} \int_0^\infty e^{-u} \cdot u^2 \cdot du$

$= \frac{1}{2} \int_0^\infty e^{-u} \cdot u^{3-1} \cdot du = \int_0^\infty e^{-t^2} \cdot t^5 \cdot dt$

$= \frac{1}{2} \Gamma(3) = \frac{2}{2} = 1$ Ans.

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Beta Function

Notation: $\beta(m,n)$

Formula: $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$

$Re(m) > 0$
$Re(n) > 0$

Relation b/w β and Γ function

$= \beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} \cdot dx$

$= \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

Q. Evaluate the integral $\int_0^2 x^3 \left(1-\frac{x}{2}\right)^4 \cdot dx$

= By using β funcn. formula,

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$

$Re(m) > 0$
$Re(n) > 0$

By using substitution method,

Let $u = \frac{x}{2}$

$du = \frac{1}{2} \cdot dx \quad \Rightarrow dx = 2du$

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When $x=0$, then $u=0$
$x=2$, then $u=1$

$$\beta(m,n) = \int_{x=0}^{x=2} x^3 \cdot \left(1-\frac{x}{2}\right)^4 \cdot dx$$

$$= \int_{u=0}^{u=1} (2u)^3 \cdot (1-u)^4 \cdot 2\,du$$

$$= 16\int_0^1 u^3 \cdot (1-u)^4 \cdot du$$

$$= 16\int_0^1 u^{4-1}\,(1-u)^{5-1}\,du$$

Using formula $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$= \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

Then above integral becomes,

$$\beta(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(4+5)}$$

$$= \frac{\cancel{3!}\,4! \times 16}{\cancel{8!}\,\cancel{9}\times 8\times 7\times \cancel{6}\times 5} = \frac{2}{\cancel{8}\times 7\times 5}\cdot\frac{\cancel{16}}{} $$

$$= \boxed{\frac{2}{35}}$$

Q Evaluate the integral

$$\int_0^1 x^4(1-\sqrt{x})^5 \cdot dx$$

By using substitution method,

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$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

Let $\sqrt{x} = u \Rightarrow x = u^2$

$$\frac{1}{2}x^{-1/2}\,dx = du$$

$$dx = 2(x)^{1/2} \cdot du$$

$$dx = 2u \cdot du$$

When $x=1 \Rightarrow u=1$
$x=0 \Rightarrow u=0$

$$\cancel{\beta(m,n) = \int_0^1 (u^2)^{m-1}(1-u^2)^{n-1}}$$

$$= \int_{x=0}^{x=1} x^4(1-\sqrt{x})^5 \cdot dx = \int_{u=0}^{u=1} (u^2)^4(1-u)^5 \cdot 2u \cdot du$$

$$= 2\int_{u=0}^{u=1} u^9 \cdot (1-u)^5 \cdot du = 2\int_{u=0}^{u=1} u^{10-1}(1-u)^{6-1} \cdot du$$

By using $\beta(m,n)$ formula

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{2\,\Gamma(10)\,\Gamma(6)}{\Gamma(10+6)}$$

$$= \frac{\cancel{2}\times\cancel{9!}\times\cancel{5!}\;\cancel{8}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{\cancel{15!}\;\cancel{15}\times\cancel{14}\times 13\times\cancel{12}\times 11\times 10}$$
3 7 4 2

$$= \frac{1}{13\times\cancel{7}\times 2\times 11\times 10}$$

Q. Evaluate the integral $\cancel{\int_0^1 (1-x^3)^9}$

$$\int_0^1 (1-x^3)^4 \cdot dx$$

= By using substitution method

$x^3 = \cancel{t} = u$

$x = u^{1/3}$

$dx = \frac{1}{3} u^{-2/3} \cdot du$

When $x=0 \Rightarrow u=0$

When $x=1 \Rightarrow u=1$

$$= \int_0^1 (1-x^3)^4 \cdot dx = \int_{u=0}^{u=1} (1-u)^4 \cdot \frac{1}{3} u^{-2/3} \cdot du$$

$$= \frac{1}{3}\int_0^1 u^{-\frac{2}{3}+1-1} (1-u)^4 \cdot du$$

$$= \frac{1}{3}\int_{u=0}^{u=1} u^{\frac{1}{3}-1} (1-u)^{5-1} \cdot du$$

By using β func^n formula,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma(5)}{\Gamma\left(\frac{1}{3}+5\right)}$$

$$= \frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma(5)}{\Gamma\left(\frac{16}{3}\right)}$$

Q. Prove that $\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \cdot d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

Proof:

= By using β func^n formula,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

= Let us suppose $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

When $x=0$, then $0 = \sin^2\theta$

$\theta = 0$

When $x=1$, then $1 = \sin^2\theta$

$\theta = \frac{\pi}{2}$

$$= \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

$$= \int_{\theta=0}^{\theta=\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \cdot \sin\theta\cos\theta\, d\theta$$

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$$= 2\int_{\theta=0}^{\theta=\pi/2} (\sin^{2m-2}\theta)(\cos^{2n-2}\theta)\sin\theta\cdot\cos\theta\cdot d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi/2} (\sin^{2m-2+1}\theta)(\cos^{2n-2+1}\theta)\cdot d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi/2} (\sin^{2m-1}\theta)(\cos^{2n-1}\theta)\cdot d\theta$$

Let $2m-1=p$

$2n-1=q$

$m=\frac{p+1}{2}$

$n=\frac{q+1}{2}$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = 2\int_{\theta=0}^{\theta=\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = 2\int_{\theta=0}^{\theta=\pi/2}\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\cdot d\theta$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} = \int_{\theta=0}^{\theta=\pi/2}\sin^p\theta\cos^q\theta\cdot d\theta$$

Formula: $\int_0^1 x^m(\log x)^n\cdot dx = \frac{(-1)^n\Gamma(n+1)}{(m+1)^{n+1}}$

Ex- Evaluate the integral ~~Do proof Assign~~

$$\int_0^1 (x\cdot\log x)^4\cdot dx = ?$$

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Assignment: ⇒ Prove this formula:

Q. ⇒ Sol^n Given $\int_0^1 (x\cdot\log x)^4\cdot dx = \int_0^1 x^4(\log x)^4 dx$

$$= \int_0^1 x^m(\log x)^n\cdot dx$$

$m=4,\ n=4$

$$= \frac{(-1)^4}{(4+1)^{4+1}}\Gamma(4+1) = \frac{\Gamma(5)}{(5)^5} = \frac{4!}{(5)^5} = \frac{4\times3\times2}{(5)^5} = \frac{24}{5^5}$$

Q. Evaluate the integral

$$\int_0^\infty e^{-3\sqrt{x}}\cdot x^{3/2}\cdot dx$$

= By using substitution method

By using Gamma formula

$$\Gamma(n) = \int_0^\infty e^{-t}\,t^{n-1}\cdot dt$$

By using subs. method

Let $3\sqrt{x}=u$

$\sqrt{x}=\frac{u}{3}$

$x=\left(\frac{u}{3}\right)^2$

$dx=\frac{2}{9}u\,du$

When $x=0$, then $u=0$

" $x=\infty$, then $u=\infty$

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$$= \int_{u=0}^{u=\infty} e^{-u}\left(\frac{1}{9}u^2\right)^{3/2}\cdot\frac{2}{9}u\cdot du$$

$$= \frac{1}{9^{3/2}}\cdot\frac{2}{9}\int_{u=0}^{u=\infty} e^{-u}\cdot u^4\, du$$

$$= \frac{1}{9^{3/2+1}}\cdot 2\quad \Gamma(5) = \frac{2}{(3^{2})^{5/2}}\cdot 4! = \frac{2\times4\times3\times2}{3^5} = \frac{24}{3^4}$$

Multiple Integral

Q. $\int_0^\infty\int_0^\infty e^{-x^2+y}\, x\cdot dx\,dy$

$$= \int_0^\infty\int_0^\infty e^{-x^2}\cdot e^{y}\cdot x\cdot dx\,dy$$

$$= \int_0^\infty e^{y}\int_0^\infty \boxed{e^{-x^2}x\cdot dx}\cdot dy$$

$$= \int_0^\infty e^{y}\int_0^\infty \boxed{e^{-x^2}\cdot x\cdot dx}\cdot dy$$

By using substitution method

$x^2 = t$ when $x=0, t=0$

$x = t^{1/2}$ 〃 $x=\infty, t=\infty$

$dx = \frac{1}{2}t^{-1/2}\cdot dt$

$$= \int_{y=0}^{y=\infty} e^{y}\int_{t=0}^{t=\infty}\left[e^{-t}\cdot t^{1/2}\cdot\frac{1}{2}\cdot t^{-1/2}\cdot dt\right]\cdot dy$$

n-tuples → n-dimensions

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$$= \int_{y=0}^{y=\infty}\frac{1}{2}e^{y}\cdot\left[\int_{t=0}^{t=\infty} e^{-t}\cdot dt\right]\cdot dy$$

$$= \int_{y=0}^{y=\infty}\frac{1}{2}e^{y}\left[\int_{t=0}^{t=\infty} e^{-t}\, t^{1-1}\cdot dt\right]\cdot dy$$

$$= \frac{1}{2}\int_{y=0}^{y=\infty}\frac{1}{2}e^{y}dy = \infty$$

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Q. $\int_0^1\int_0^1 \frac{dx\cdot dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$$= \int_{y=0}^{y=1}\int_{x=0}^{x=1}\frac{dx\,dy}{\sqrt{1-x^2}\cdot\sqrt{1-y^2}} = \int_{y=0}^{y=1}\frac{dy}{\sqrt{1-y^2}}\cdot\int_{x=0}^{x=1}\frac{dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}y\right]_{y=0}^{y=1}\cdot\left[\sin^{-1}x\right]_{x=0}^{x=1} = \left(\frac{\pi}{2}-0\right)\cdot\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi^2}{4}$$

Q. Evaluate the integral $\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy$

$$= \int\sqrt{a^2-x^2}\cdot dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \int_{y=0}^{y=a}\left[\frac{x\sqrt{a^2-y^2-x^2}}{2}+\frac{(a^2-y^2)}{2}\cdot\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_{x=0}^{x=\sqrt{a^2-y^2}}dy$$

$$= \int_{y=0}^{y=a}\left(\frac{\sqrt{a^2-y^2}\sqrt{a^2-y^2-(\sqrt{a^2-y^2})^2}}{2}+\frac{a^2-y^2}{2}\sin^{-1}\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}} - 0 + 0\right)$$

"When we bound region, ... area"
Region → n dimensions से आता है

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$$= \int_{y=0}^{y=a} \left[0 + \frac{a^2-y^2}{2} \sin^{-1} 1 - 0 \right] dy$$

$$= \int_{y=0}^{y=a} \left(\frac{a^2-y^2}{2} \sin^{-1} 1 \right) dy$$

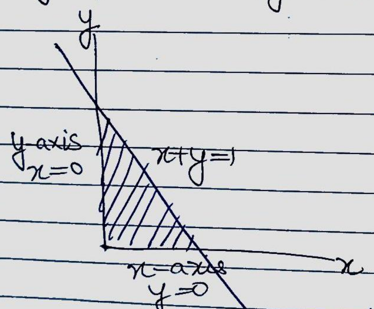
$$= \frac{\pi}{4} \int_{y=0}^{y=a} \left(a^2 - y^2 \right) dy$$

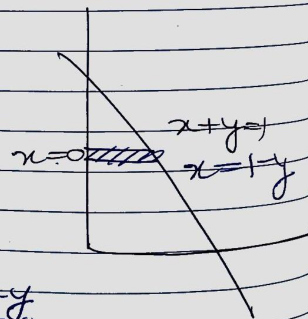
$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left(a^3 - \frac{a^3}{3} - 0 \right)$$

$$= \frac{\pi}{4_2} \cdot \frac{\cancel{2}a^3}{3} = \frac{\pi a^3}{6}$$

Q. Evaluate the integral $\iint_R xy\,dx\,dy$ where R is the region bounded by $x+y=1$, $x=0$, $y=0$.

⇒





$$= \iint_R xy\,dx\,dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} xy\,dx\,dy$$

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$$= \int_{y=0}^{y=1} \left[\frac{x^2}{2} \right]_{x=0}^{x=1-y} \cdot dy$$

$$= \int_{y=0}^{y=1} y \cdot \left[\frac{x^2}{2} \right]_{x=0}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1} y \left[\frac{(1-y)^2}{2} - \frac{0}{2} \right] \cdot dy$$

$$= \frac{1}{2} \int_{y=0}^{y=1} y \left(1 + y^2 - 2y \right) dy$$

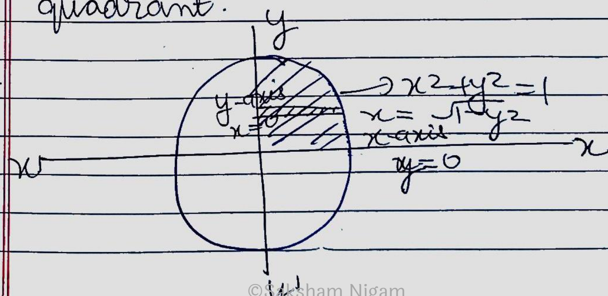
$$= \frac{1}{2} \int_{y=0}^{y=1} \left[y + y^3 - 2y^2 \right] dy$$

$$= \frac{1}{2} \left(\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3} \right)_0^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) - 0 \right] = \frac{1}{2} \left[\frac{6+3-8}{12} \right] = \frac{1}{24}$$

Q. Evaluate the integral $\iint_R x^2 y^2 \cdot dx \cdot dy$; where R is the region bounded by $x^2+y^2=1$ and $x=0$, $y=0$ in first quadrant.

Soln.



$$= \int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}} x^2\cdot y^2\cdot dx\,dy$$

$$= \int_{y=0}^{y=1} y^2\cdot dy \int_{x=0}^{x=\sqrt{1-y^2}} \cdot x^2 . dx$$

$$= \int_{y=0}^{y=1} y^2 . dy \left[\frac{x^3}{3}\right]_0^{\sqrt{1-y^2}}$$

$$= \int_{y=0}^{y=1} y^2 \frac{(1-y^2)^{3/2}}{3} . dy$$

$$= \frac{1}{3}\int_{y=0}^{y=1} y^2 (1-y^2)^{3/2} . dy$$

$$= \frac{1}{3}\int_{y=0}^{y=1} (y^2)^{2-1} (1-y^2)^{\frac{3}{2}+1-1} dy$$

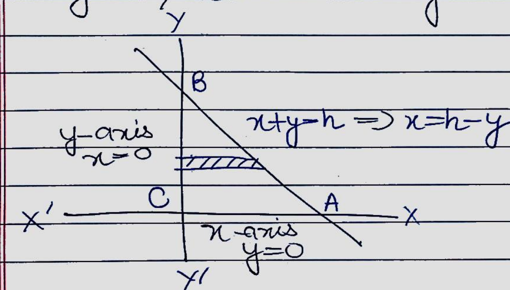
$$= \frac{1}{3}\int_{y=0}^{y=1} (y^2)^{2-1} (1-y^2)^{\frac{5}{2}-1} . dy$$

$$= \frac{1}{3}\,\frac{\Gamma(2)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(2+\frac{5}{2}\right)}$$

$$= \frac{\frac{1}{3}\cdot 1\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{1}{2}}\cancel{\sqrt{\pi}}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{1}{2}}\cdot\cancel{\sqrt{\pi}}} = \frac{1\times2\times2}{3\times7\times5} = \frac{4}{105}$$

Q. Evaluate the integral $\iint_R x^{m-1} y^{n-1} dx\,dy$ where R is the region bounded by $x+y=h$, $x=0$ and $y=0$.

=



$$\iint_R x^{m-1} y^{n-1} dx\,dy = \int_{y=0}^{h}\int_{x=0}^{x=h-y} x^{m-1} y^{n-1} dx\,dy$$

$$= \int_{y=0}^{y=h} y^{n-1}\left[\int_{x=0}^{x=h-y} x^{m-1} dx\right] dy$$

$$= \int_{y=0}^{y=h} y^{n-1}\left[\frac{x^m}{m}\right]_{x=0}^{x=h-y} dy$$

$$= \int_{y=0}^{y=h} y^{n-1}\left[\frac{(h-y)^m}{m} - \frac{0^m}{m}\right] dy$$

$$= \frac{1}{m}\int_{y=0}^{y=h} y^{n-1} (h-y)^m . dy$$

$$= \frac{1}{m}\int_{y=0}^{y=\cancel{g}=h} y^{n-1} (h-y)^m . dy$$

Let $y = ht \Rightarrow dy = h\,dt$

When $y=0 \Rightarrow t=0$

" $y=h \Rightarrow t=1$

$$= \frac{1}{m}\int_{y=0}^{y=h} \cancel{y^{m-1}}\, y^{n-1}(h-y)^{m}\cdot dy$$

$$= \cancel{\frac{1}{m}\int_{y=0}^{y=h}}\ \frac{1}{m}\int_{t=0}^{t=1} h^{n-1}t^{n-1}h^{m}(1-t)^{m}\times h\,dt$$

$$= \frac{1}{m}\int_{t=0}^{t=1} h^{m+n}t^{n-1}(1-t)^{m+1-1}\,dt$$

$$= \frac{1}{m}h^{m+n}\int_{0}^{1} t^{n-1}(1-t)^{m+1-1}\,dt$$

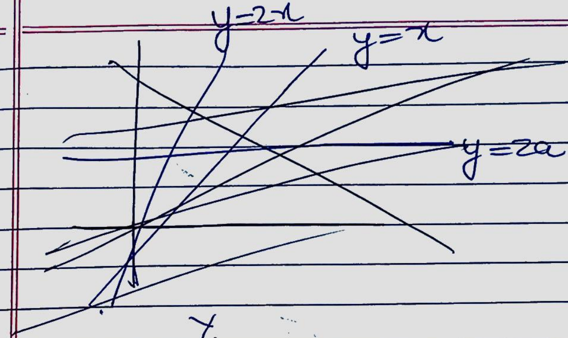
Using β-funcⁿ formula

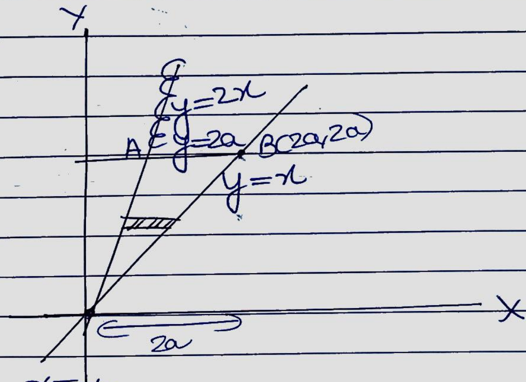
$$= \frac{1}{m}h^{n+m}\frac{\Gamma(n)\Gamma(m+1)}{\Gamma(n+m+1)}$$

$$= \frac{1}{\cancel{m}}\cdot h^{n+m}\frac{\Gamma(n)\cdot \cancel{m}\,\Gamma(m)}{\Gamma(n+m+1)}$$

$$= h^{n+m}\cdot\frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m+1)}$$

Q. Evaluate the integral $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by $y=x$, $y=2x$ and $y=2a$





$$= \int_{y=0}^{y=2a}\int_{x=y/2}^{x=y}(x^2+y^2)\cdot dx\cdot dy$$

$$= \int_{y=0}^{y=2a}\left[\frac{x^3}{3}+y^2x\right]_{x=y/2}^{x=y}dy$$

$$= \int_{y=0}^{y=2a}\left(\left(\frac{y^3}{3}+y^3\right)-\left(\frac{y^3}{48}+\frac{y^3}{2}\right)\right)$$

$$= \int_{y=0}^{y=2a}\left(\frac{16y^3+48y^3-y^3-24y^3}{48}\right)$$

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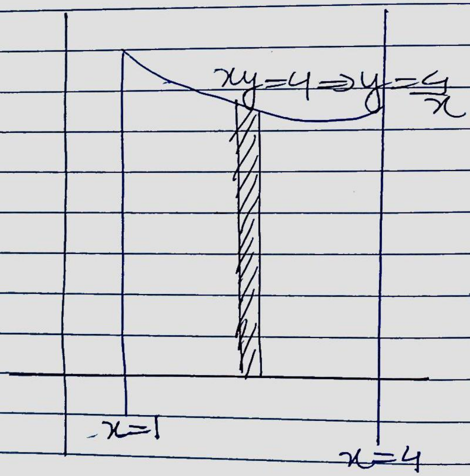
$$= \int_{y=0}^{y=2a} \frac{\cancel{39}^{13} y^3}{\cancel{48}_{16}}$$

$=$

Q. Evaluate the integral $\iint_R xy(1-x)\,dx\,dy$ where R is the region bounded by $xy=4$ and $x=1$ and $x=4$



$$\iint_R xy(1-x)\,dx\,dy$$

$$= \int_{x=1}^{x=4}\int_{y=0}^{y=\frac{4}{x}} xy(1-x)\cdot dx\,dy$$

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$$= \int_{x=1}^{x=4} dx \int_{y=0}^{y=\frac{4}{x}} (xy - x^2y)\,dy$$

$$= \int_{x=1}^{x=4} \left[\frac{xy^2}{2} - \frac{x^2y^2}{2}\right]_0^{4/x}$$

$$= \int_{x=1}^{x=4}$$

$$= \int_{x=1}^{x=4} (x)(1-x) \left[\frac{y^2}{2}\right]_{y=0}^{y=\frac{4}{x}} \cdot dx$$

$$= \int_{x=1}^{x=4} x(1-x)\left[\frac{\left(\frac{4}{x}\right)^2}{2} - \frac{0}{2}\right] dx$$

$$= \int_{x=1}^{x=4} x(1-x)\left(\frac{\cancel{16}\,8}{x^2\cdot\cancel{2}}\right) dx = 8\int_{x=1}^{x=4} \frac{1-x}{x}\,dx$$

$$= 8\int_{x=1}^{x=4} \left(\frac{1}{x} - 1\right) dx \quad 8$$

$$= 8\left[\ln x - x\right]_1^4 = 8\left[(\ln 4 - 4) - (\ln 1 - 1)\right]$$

$$= 8\left[\ln 4 - 3\right] = 8\cdot \ln 2^2 \cdot 24$$

$$= 8\times 2\ln 2 - 24$$

$$= 16\ln 2 - 24$$

Assignment

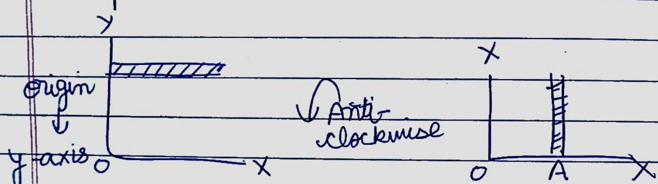
Q. Evaluate the $\iint_R \sqrt{xy(1-x-y)}\,dx\,dy$

where R is the region bounded by $x=y$, $x=1$ and $y=2$

\# Change the order of integration

\# $\int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\cdot dx\,dy = \int_{x=g}^{x=h}\int_{y=e}^{y=f} f(x,y)\,dx\,dy$

Strip



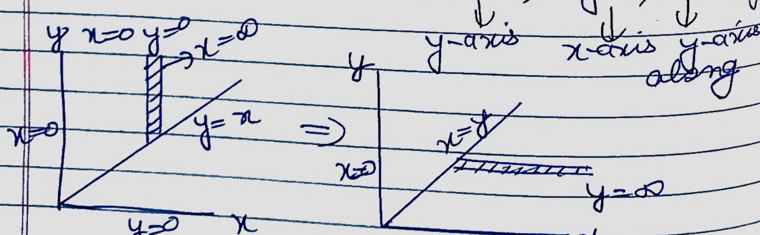
eg. Change the order of integration

$\int_{x=0}^{x=\infty}\int_{y=0}^{y=x} e^{-xy}\,dy\,dx = ?$

$\Rightarrow$ Given $\int_{x=0}^{x=\infty}\int_{y=0}^{y=x} e^{-xy}\,dy\,dx$ where

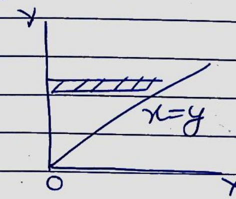
$x=0$, $y=0$, $x=\infty$, $y=x$

↓ y-axis, ↓ x-axis, ↓ y-axis along



Ex- $\int_{x=0}^{x=\infty}\int_{y=0}^{y=x} e^{-xy}\,dy\,dx$

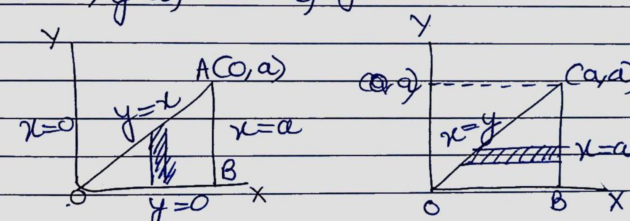
Soln $y=0$, $y=x$, $~~z~~=0$, $x=\infty$



Q. Change the order of the integration

$\int_{x=0}^{x=a}\int_{y=0}^{y=x} f(x,y)\cdot dy\cdot dx$

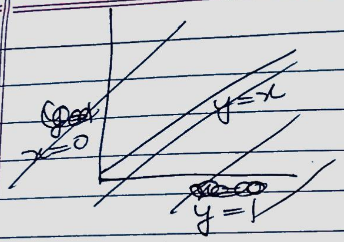
Soln. $x=0$, $y=0$, $x=a$, $y=x$

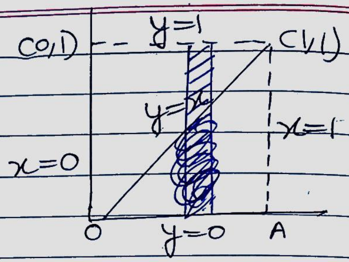


Q. Change the order of the integration

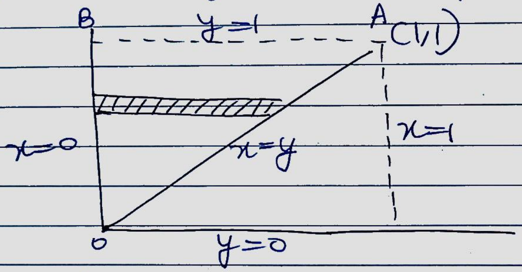
$\int_0^1\int_x^1 \sin(y^2)\cdot dy\cdot dx = ?$

$=$ $y=x$, $y=$ ___ $x=$ ___ $x=0$





If we change the strip



$$\int_{x=0}^{y=1}\int_{y=x}^{y=1} \sin(y^2)\cdot dy\,dx \qquad \int_{y=0}^{y=1}\int_{x=0}^{x=y} \sin(y^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \sin y^2 \,[x]_{x=0}^{x=y}$$

$$= \int_{y=0}^{y=1} y\sin(y^2)\,dy$$

When $y=0 \Rightarrow t=0$
$y=1 \Rightarrow t=1$

Let $y^2 = t$
$2y\,dy = dt$
$y\,dy = \frac{dt}{2}$

$$= \frac{1}{2}\int_{t=0}^{t=1} \sin t\cdot dt$$

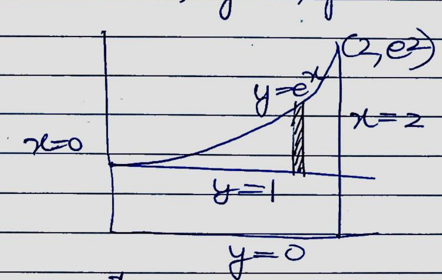
$$= -\frac{1}{2}\left[\cos t\right]_{t=0}^{t=1}$$

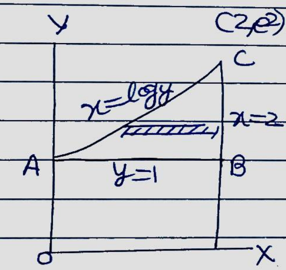
$$= -\frac{1}{2}\left[\cos 1 - \cos 0\right]$$

Q. Change the order of the integration

$$\int_{x=0}^{x=2}\int_{y=1}^{y=e^x} f(x,y)\cdot dx$$

$= x=0,\ x=2,\ y=1,\ y=e^x$



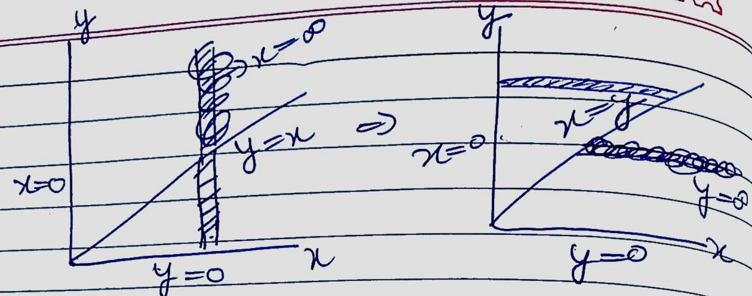


$$= \int_{x=0}^{x=2}\int_{y=1}^{y=e^x} f(x,y)\cdot dx = \int_{y=1}^{y=e^2}\int_{x=\ln y}^{x=2} f(x,y)\,dx\,dy$$

Q. Change the order of integration

$$\int_0^\infty\int_0^x e^{-xy}\, y\,dy\,dx$$

Soln $x=0,\ x=\infty,\ y=0,\ y=x$



$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=y} e^{-xy}\, y\, dx\, dy$$

change of variables

Transformation

- Plane to Plane (Jacobian)
  - $\boxed{(x,y)} \rightarrow \boxed{(u,v)}$ coordinate
- Space to space (Some particular operation).
  - $\boxed{t\text{-space}} \rightarrow \boxed{s\text{-space}}$ Parameter

Coordinates

- Cartesian (x,y)
- Polar (r,θ)
  $\Downarrow$
  $x = r\cos\theta$
  $y = r\sin\theta$
- Cylindrical (r,θ,z)
  $x = r\cos\theta$
  $y = r\sin\theta$
  $z = z$
- Spherical (r,θ,ϕ)
  $x = r\sin\theta\cos\phi$
  $y = r\sin\theta\sin\phi$
  $z = r\cos\theta$

$$\iint dx\,dy = J\,dx\,dy$$

i) Polar co-ordinate: $x = r\cos\theta$
$y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = J \qquad dx\,dy = J\,dr\,d\theta$$
$$dx\,dy = r\,dr\,d\theta$$

ii) Cartesian $\rightarrow$ cylindrical

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \end{vmatrix}$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \left| r\cos^2\theta + r\sin^2\theta = r \right.$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = J \qquad dx\,dy\,dz = J\,dr\,d\theta\,dz$$
$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

(iii) Cartesian to spherical

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = J =$$

$$dx\,dy\,dz = J\,dr\,d\theta\,d\phi$$
$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Ex- By using change of variables, evaluate the integral

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\cdot dx\,dy$$

Sol^n By using polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = J\,dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

(1) Polar eq^n of the the straight line $x=0$

$$x = 0$$
$$r\cos\theta = 0$$
$$\cos\theta = 0$$
$$\theta = \cos^{-1}0$$
$$\theta = \frac{\pi}{2}$$

(2) Polar eq^n of the circle

$$x = \sqrt{a^2-y^2}$$
$$x^2 = a^2-y^2$$
$$x^2+y^2 = a^2$$
$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$
$$r^2 = a^2$$

(3) Equation of circle

$$(x-a)^2+(y-b)^2 = d^2$$
$$x = \sqrt{a^2-y^2}$$
$$x^2+y^2 = a^2$$
$$(x-0)^2+(y-0)^2 = a^2$$

Limit of integration becomes:

$$r=0 \text{ to } r=a$$
$$\theta=0 \text{ to } \theta=\frac{\pi}{2}$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} (\cancel{r^2\cos^2\theta + r^2\sin^2\theta})\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=a} r^3 \cdot dr\, d\theta = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=a} r^3 dr$$

$$= [\theta]_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{r=a}$$

$$= \frac{\pi}{2} \frac{a^4}{4} = \frac{\pi a^4}{8} \text{ Ans.}$$

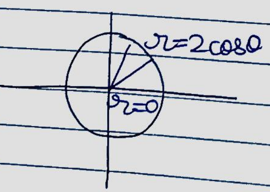
e.g. Evaluate the integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ by using change of variables method

By using change of variables method.
By using polar coordinate, $x = r\cos\theta$
$y = r\sin\theta$

$dx\,dy = J\,dr\,d\theta$, $J = r$
$dx\,dy = r\,dr\,d\theta$

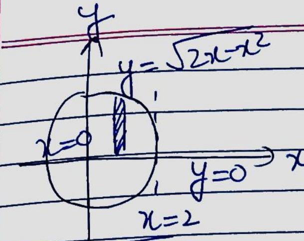
$$\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{2x-x^2}} \frac{x \cdot dx\,dy}{\sqrt{x^2+y^2}}$$

Given $x=0$, $x=2$, $y=0$, $y=\sqrt{2x-x^2}$



$y = \sqrt{2x-x^2}$
$y^2 + x^2 = 2x$
$r^2(\cos^2\theta + \sin^2\theta) = 2r\cos\theta$
$r = 2\cos\theta$

Kindly recheck, there may be some errors



$x=0$, $x=2$
$y=0$, $y=\sqrt{2x-x^2}$
r varies from 0 to $2\cos\theta$
θ varies from 0 to $\frac{\pi}{2}$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta\, r\,dr\,d\theta}{r} = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2\cos\theta} r\cos\theta\, dr\,d\theta$$

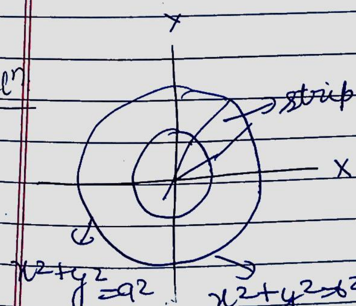
$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=2\cos\theta} d\theta = \int_0^{\pi/2} \cos\theta \left[\frac{2\cos^2\theta}{2} - \frac{0}{2}\right] d\theta$$

$$= 2\int_0^{\pi/2} \cos^3 d\theta = 2\int_0^{\pi/2} \cos^3\theta \sin^0\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\, \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$= \frac{\cancel{2} \times \sqrt{2} \times \Gamma\frac{1}{2}}{\cancel{2}\, \Gamma\frac{5}{2}} = \frac{4}{3}$$

e.g. Evaluate integral $\iint \frac{x^2y^2}{x^2+y^2} dx\,dy$, $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $a>b>0$.

Sol^n



$$\iint \frac{x^2y^2}{x^2+y^2} dx\,dy = \int_0 \int \frac{r^2\cos^2\theta\, r^2\sin^2\theta\, dr\,d\theta\; r}{r^2}$$

$x^2+y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$
$x^2+y^2 = b^2 \Rightarrow r^2 = b^2 \Rightarrow r = b$
$\theta = 0$ to $\theta = \frac{\pi}{2}$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=b}^{r=a} r^3 \cdot \cos^2\theta \sin^2\theta \cdot dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_b^a \cos^2\theta \sin^2\theta \, d\theta = \frac{a^4-b^4}{4}\int_0^{\pi/2} \cos^2\theta\sin^2\theta\, d\theta$$

$$= \frac{a^4-b^4}{4}\;\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 3} = \frac{a^4-b^4}{4}\cdot\frac{1}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{64}\left(a^4-b^4\right)$$

e.g. Evaluate the integral $\int_0^1\int_0^x (x+y)\,dy\,dx$ by using polar coordinates.

$$\Rightarrow \int_0^{\pi/4}\int_0^{\sec\theta} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = J\,dr\,d\theta$

$dx\,dy = r\,dr\,d\theta$

$y = x$

$r\sin\theta = r\cos\theta$

$\tan\theta = 1$

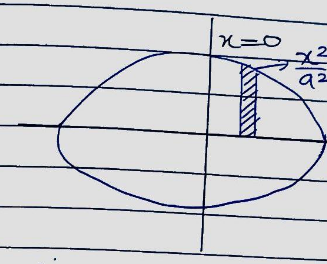
$\theta = \frac{\pi}{4}$

Kindly recheck, there may be some errors

7/10/24

Area and volume by the Double Integral

Ex-1) Find the area of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$y = b\sqrt{1-x^2/a^2}$

For x-axis (y=0), $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{x^2}{a^2}=1$

$x = a$ ($\neq -a$, For first quadrant) by me

area $= \iint dx\,dy$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b\sqrt{1-x^2/a^2}} dy\,dx = \int_{x=0}^{x=a}[y]_{y=0}^{y=b\sqrt{1-x^2/a^2}}\,dx$$

$$= \int_{x=0}^{x=a}\left[b\sqrt{1-\frac{x^2}{a^2}}-0\right]dx = \frac{b}{a}\int_{x=0}^{x=a}\sqrt{a^2-x^2}\,dx$$

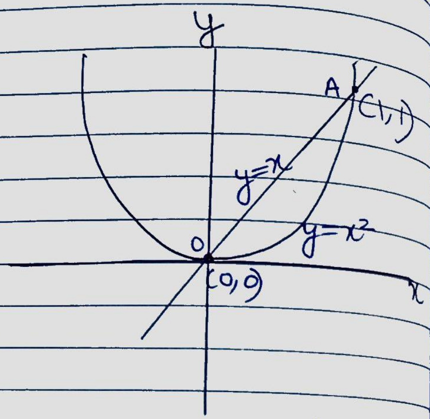
$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\cancel{\frac{a}{2}\sqrt{a^2-a^2}}+\frac{a^2}{2}\sin^{-1}\frac{a}{a} - 0 + \frac{a^2}{2}\sin^{-1}\frac{0}{a}\right]$$

$$= \frac{b}{\cancel{a}}\cdot\frac{a^{\cancel{2}}}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}$$

Kindly recheck, there may be some errors

Q. Find the area b/w the curve $y=x^2$ and $y=x$.



Area $= \iint dy\,dx$

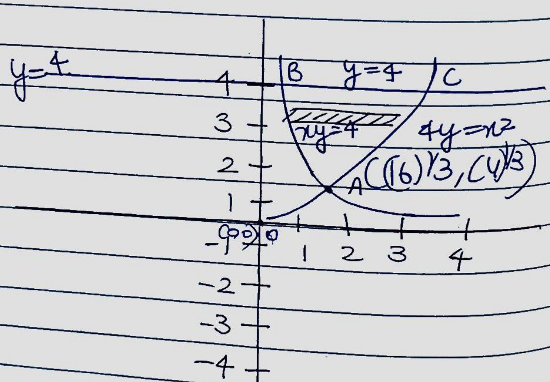
$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} dy\,dx$$

$$= \int_{x=0}^{x=1} \left[y\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} (x-x^2)\cdot dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

Q. Find the area b/w the curve $xy=4$, $4y=x^2$ and $y=4$.



$$= \int_{y=1}^{y=4}\int_{x=\frac{4}{y}}^{x=2\sqrt{y}} dx\,dy$$

$$= \int_{y=1}^{y=4}\left(2\sqrt{y} - \frac{4}{y}\right)dy = \left[\frac{2y^{3/2}}{3}\cdot 2 - 4\ln y\right]_{y=1}^{y=4}$$

$$= \left[\frac{4}{3}y^{3/2} - 4\ln y\right]_{y=1}^{y=4}$$

$$= \frac{4}{3}\cdot(2)^{2\cdot\frac{3}{2}} - 4\ln 4 - \frac{4}{3} + 4\ln 1$$

$$= \frac{4}{3}\cdot 8 - \frac{4}{3} + 4\left(\ln\frac{1}{4}\right)$$

$$= \frac{7\cdot 4}{3} + 4\ln\left(\frac{1}{4}\right) = \frac{28}{3} + 4\ln\left(\frac{1}{4}\right)$$

$$\frac{28}{3} - 4\ln 4$$

$$= \frac{28}{3} - 4\ln 2$$

Q. Find the area b/w the curve $y^2(2-x)=x^3$.

Solⁿ. ① Symmetric

यदि x के सारे power even है तो y को along symmetric होगा and vice versa

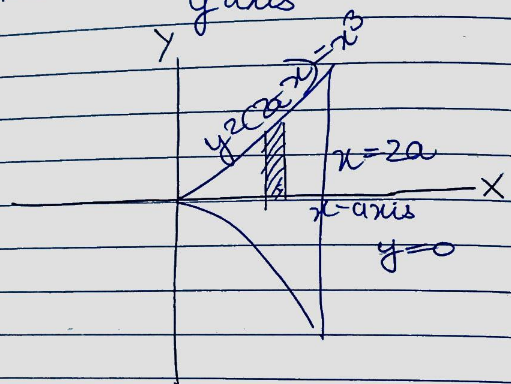
② Tangent: $y=0$

Cusp - [illegible] node - चौराहा

Kindly recheck, there may be some errors

We never say $\cos^{-2}$ | or $\cos^{-3}$ | ...
because the part will remain [illegible]
arc size of curve will remain same

③ Asymptote : $x^3 = 0$
y axis



By using double integration

$$y^2(2a-x) = x^3$$

$$y^2 = \frac{x^3}{2a-x} \Rightarrow y = \frac{x^{3/2}}{\sqrt{2a-x}}$$

$$\text{Area} = \iint dy\,dx = \int_{x=0}^{x=2a} dx \int_{y=0}^{y=\frac{x^{3/2}}{\sqrt{2a-x}}} dy$$

$$= \int_{x=0}^{x=2a} dx \left[y \right]_0^{\frac{x^{3/2}}{\sqrt{2a-x}}}$$

$$= \int_{x=0}^{x=2a} \frac{x^{3/2}\,dx}{\sqrt{2a-x}} =$$

When $x=0$, $\theta = 0$
When $x = 2a$, $\theta = \pi/2$

Let $x = 2a\sin^2\theta$
$dx = 2a\,2\sin\theta\cos\theta\,d\theta$
$dx = 2a\sin 2\theta$

$$= \int_{x=0}^{x=2a} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{2a-2a\sin^2\theta}} \cdot 2a\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{2a}\sqrt{\cos^2\theta}} (2a\cdot 2\cdot\sin\theta\cos\theta\cdot d\theta)$$

$$\int_{\theta=0}^{\theta=\pi/2} \sin^p\theta \sin^q\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= \frac{(2a)^{3/2}\,4a}{\sqrt{2a}} \int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= \frac{(2a)^{3/2}}{(2a)^{1/2}} \cdot 4a \int_{\theta=0}^{\theta=\pi/2} \sin^4\theta \cdot \cos^0\theta\,d\theta$$

$$= \left(\frac{(2a)^{\cancel{3}\,2}}{\cancel{2a}}\right)^{1/\cancel{2}} \cdot (4a)$$

$$= \frac{8a^2 \cdot \Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}$$

$$= \frac{8a^2 \cdot \Gamma\left(\frac{5}{2}\right)\cdot\sqrt{\pi}}{2\left(\Gamma\frac{\cancel{6}\,3}{\cancel{2}}\right)} = \frac{\cancel{8}^{\,2}a^2 \cdot \frac{\cancel{5}}{\cancel{2}}\cdot\frac{3}{2}\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot\cancel{2}}$$

$$= \frac{\cancel{2}a^2}{\cancel{4}} \cdot \pi = \frac{\pi a^2}{2}$$

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Volume by using Double Integral

Formula for volume $\iint_V z\,dx\cdot dy$

Ex- Calculate the volume of the pyramid bounded by the coordinate axes and the plane $x + 2y + 3z = 6$.

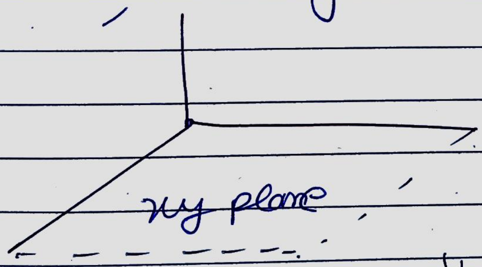
$=$ $z - x + 2y + 3z = 6$

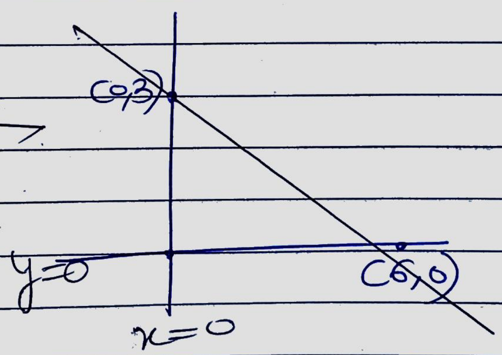
$f(x,y) = x + 2y + 3z = 6$

$$3z = 6 - x - 2y$$

$$z = \frac{6 - x - 2y}{3}$$

For x-axis For y-axis



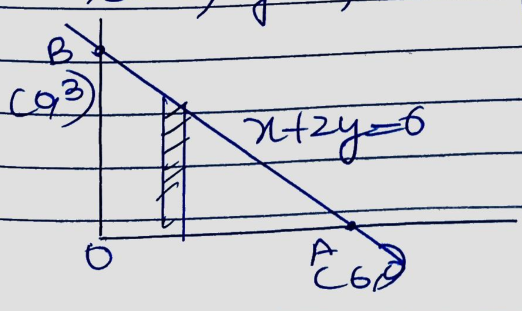


$z = 0$

$x + 2y = 6$

According to question, pyramid

$x = 0, \ y = 0 \quad x + 2y = 6$



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Volume $= \iint z\,dy\,dx$

$$= \int_{x=0}^{x=6}\int_{y=0}^{y=6-x} \left(\frac{6-x-2y}{3}\right) dy\,dx$$

$$= \int_{x=0}^{x=6} dx \left[2y - xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_{y=0}^{y=6-x}$$

$$= \int_{x=0}^{x=6} dx \left[2(6-x) - x(6-x) - (6-x)^2\right]$$

$$= \int_{x=0}^{x=6} dx \left[12 - 2\cancel{x} - 6\cancel{x} + \cancel{x^2} - 36 \cancel{-x^2} + 12\cancel{x}\right]$$

$$= \int_{x=0}^{x=6} (4x - 24) = \int_{x=0}^{x=6} 4(x-6)$$

$$= 4\left[\frac{x^2}{2} - 6x\right]_0^6 = 4\left[\frac{36}{2} - 36\right]$$

$$= 4(-18) = -72$$