Differential Equation

→ Homogenous Equation

$$m^2 + 6m + 6 = 0$$

→ Non-Homogenous Equation

$$m^2 + 5m + 6 = e^x.$$

Constant and Arbitrary constant

$$\frac{d}{dx} c = 0$$

Operator: To change the nature of the function.
↳ Rate of change of 'c' w.r.t x but 'c' is constant.

Constant: A function whose rate of change can be calculated is known as a variable, a function whose rate of change cannot be calculated is known as constant.

→ Arbitrary constant:
Role of arbitrary constant to decide order of differential equation.

Integration is the inverse of differentiation.

Rate of change = work done

Arbitrary constant => Family of curve
$f(x, y, c)$, c is the arbitrary constant.

→ Formation of the differential equation

Example: Construct the differential equation for the function $y(x) = c_1 e^x + c_2 e^{-x}$, where c_1 and c_2 are arbitrary constants.

Given,

$$y(x) = c_1 e^x + c_2 e^{-x}$$

$$\frac{dy}{dx}(x) = c_1 e^x - c_2 e^{-x}$$

$$\frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x}$$

$$\therefore \quad \frac{d^2y(x)}{dx^2} = y(x)$$

$$\frac{d^2y(x)}{dx^2} - y(x) = 0 .$$ Ans

Example: Construct the differential equation for the function.

$$y(x) = C_1 \sin x + C_2 \cos x$$

$\therefore$ $$y(x) = C_1 \sin x + C_2 \cos x$$

$$\frac{dy(x)}{dx} = C_1 \cos x - C_2 \sin x$$

$$\frac{d^2y(x)}{dx^2} = -C_1 \sin x - C_2 \cos x = -y$$

$\therefore$ $$\frac{d^2y(x)}{dx^2} + y = 0.$$ Ans.

Example: Construct the differential equation for the function.

$$y = cx + \frac{1}{c}, \quad c \neq 0$$

$\therefore$ $$y(x) = cx + \frac{1}{c}$$

$\Rightarrow$ $$\frac{dy(x)}{dx} = c$$

$\therefore$ $$y = \left(\frac{dy}{dx}\right)x + \frac{1}{\left(\frac{dy}{dx}\right)}$$

$\Rightarrow$ $$\frac{dy}{dx}y = \left(\frac{d^2y}{dx^2}\right)^2 x + 1$$

$\therefore$ $$x\left(\frac{d^2y}{dx^2}\right)^2 - y\frac{dy}{dx} + 1 = 0$$

Order: 1
Degree: 2

→ Wronskian Method to determine dependent and independent form of the solution of differential equation

Solution

Explicit (स्पष्ट) (بسیط)

→ $y = f(x)$

→ Single independent variable.

Implicit (अस्पष्ट) (ابسیط)

→ $z = f(x, y)$

→ More than one independent variable.

$$W(x) = \begin{vmatrix} W_1 & W_2 & W_3 \\ W_1' & W_2' & W_3' \\ W_1'' & W_2'' & W_3'' \end{vmatrix}$$

→ If $W(x) \neq 0$ (Independent)

→ If $W(x) = 0$ (Dependent)

Example : Find the nature of the given solution of differential equation ;

$y = c_1 e^{x} + c_2 e^{-x}$

$W_1(x) = e^{x}, \quad W_2(x) = e^{-x}$

$$W(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{0} - e^{0} = -2$$

Hence, solution is independent.

Example : Find the nature of the solution of the given differential equation ;

$$y = c_1 \sin x + c_2 \cos x$$

$\therefore W_1(x) = \sin x$

$W_2(x) = \cos x$

$$\therefore \quad W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Hence, solution is independent.

→ Linear differential equation with constant coefficient

Consider the differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

a_0, a_1, a_2

Constant
Linear differential equation with constant variable

variable
Linear non-homogenous differential equation.

→ Auxiliary Equation and Auxiliary Equation roots.

Auxiliary equation:

If we have:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

$\Rightarrow$ Writing

$$\frac{d}{dx} \equiv m$$

$\therefore$ The auxiliary equation is given as:

$$a_0 m^2 y + a_1 m y + a_2 y = R(x)$$

$$\therefore \underbrace{(a_0 m^2 + a_1 m + a_2)}_{\text{Auxiliary equation}} y = R(x)$$

$$\Rightarrow \quad a_0 m^2 + a_1 m + a_2 = 0$$

→ Roots
1. Real
2. Imaginary
3. Irrational

CASE 1:

If the roots of the auxiliary equation are real and distinct

$$a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + a_{n-2} \frac{d^{n-2}}{dx^{n-2}} y + \dots + a_0 y = R(x)$$

Suppose ;

$$(D^n - m_n)(D^{n-1} - m_{n-1}) \cdots (D - m) = 0$$

$\therefore \quad D - m = 0$

$$\frac{dy}{dx} - my = 0$$

$$\left(\frac{d}{dx} - m\right)y = 0$$

$$I.F = e^{\int P dx} = e^{-mx}$$

▫ If roots are real and distinct

Complementary Function $= c_1 e^{+m_1 x} + c_2 e^{+m_2 x} + \cdots + C_n e^{m_n x}$

→ Example :

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$\therefore \quad m^2y + 6my + 9y = 0$$

$$(m^2 + 6m + 9)y = 0$$

Auxiliary is given as;

$$m^2 + 6m + 9 = 0$$

$$\therefore \quad m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$m = -3, -3$$

$$\therefore \quad C.F = (C_1 + xC_2)e^{-3x}.$$

Example : $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

The auxiliary equation is given as ;

$$m^2 + 7m + 12 = 0$$

$$\therefore \quad m^2 + 4m + 3m + 12 = 0$$

$$m = -4, -3$$

As, the roots are real and distinct, the complementary function is;

$C.F = C_1 e^{-4x} + C_2 e^{-3x}$. Ans.

Example: $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0$

$\therefore$
$$m^2y + 8my + 15y = 0$$
$$(m^2 + 8m + 15)y = 0$$

Hence, the auxiliary equation is given as;
$$m^2 + 8m + 15 = 0$$
$$m^2 + 5m + 3m + 15 = 0$$
$$m = -5, -3$$

As, the roots are real and distinct. The complementary function is given as;

$$C.F = c_1e^{-5x} + c_2e^{-3x}.$$ Ans.

CASE 2:

If the roots are real and repeated

$\Rightarrow$ If $m_1 = m_2$ and $m_3, m_4, \ldots$ are real and distinct roots.

Hence,

$$\text{Complementary Function (C.F)} = (c_1 + xc_2)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}.$$

$\Rightarrow$ If $m_1 = m_2 = m_3$ and $m_4, m_5, \ldots, m_n$ are real and distinct roots.

$$\text{Complementary Function (C.F)} = (c_1 + xc_2 + x^2c_3)e^{m_1x} + c_4e^{m_4x} + \ldots + c_ne^{m_nx}.$$

Example: $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

$\therefore$
$$m^2y + 10my + 25y = 0$$
$$(m^2 + 10m + 25)y = 0$$

The auxiliary equation is given as;
$$m^2 + 10m + 25 = 0$$
$$m = -5, -5$$

$\therefore$ As, the roots are real but repeated, the complementary function is given as;

$$C.F = (c_1 + xc_2)e^{-5x}.$$ Ans.

→ Case 3 (i): When roots are imaginary

If roots of the auxiliary equation are imaginary, the complementary function is given as;

$$C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \quad ; \quad z = \alpha + i\beta$$

→ Case 3 (ii): When roots are imaginary but repeated

$$C.F = e^{\alpha x}((c_1 + xc_2)\cos \beta x + (c_3 + xc_4)\sin \beta x)$$

→ Case 4 (i): When roots are irrational

If roots of the auxiliary equation are irrational $(\alpha \pm \sqrt{\beta})$, the complementary function is given as;

$$C.F = e^{\alpha x}(c_1 \cosh \sqrt{\beta} + c_2 \sinh \sqrt{\beta}).$$

→ Case 4 (ii): When roots are irrational and repeated.

$$C.F = e^{\alpha x}((c_1 + xc_2)\cosh \sqrt{\beta} + (c_3 + xc_4)\sinh \sqrt{\beta})$$

Example: $D^4 - n^4 y = 0$

The auxiliary equation is given as;

A.E : $m^4 - n^4 = 0$

$$(m^2 - n^2)(m^2 + n^2) = 0$$

$$(m^2 + n^2)(m - n)(m + n) = 0$$

$\therefore \quad m = n, -n$ and in

Hence, $C.F = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}(c_3 \cos nx + c_4 \sin nx)$. Ans

Example: $\dfrac{d^4 y}{dx^4} + n^4 y = 0$

The auxiliary equation is given as;

A.E $= m^4 + n^4 = 0$

$$m^4 + n^4 + 2m^2n^2 - 2m^2n^2 = 0$$

$$(m^2 + n^2)^2 - 2m^2n^2 = 0$$

$$(m^2 + n^2 + 2mn)(m^2 + n^2 - 2mn) = 0$$

$$\therefore \quad m = \frac{\sqrt{2}n \pm \sqrt{2(1) - 4(n^2)}}{2} \quad , \quad m = \frac{-\sqrt{2}n \pm \sqrt{2 - 4(n^2)}}{2}$$

$$= \frac{\sqrt{2} \pm \sqrt{2(1 - 2n^2)}}{2} \qquad m = -\frac{n}{\sqrt{2}} \pm \frac{\sqrt{1 - 2n^2}}{\sqrt{2}}$$

$$= \frac{}{\sqrt{2}} \pm \frac{\sqrt{1 - 2n^2}}{\sqrt{2}}$$

Complementary function: $e^{n/\sqrt{2}\,x}\left(c_1 \cosh\sqrt{\frac{1-2n^2}{2}} + c_2 \sinh\sqrt{\frac{1-2n^2}{2}}\right) +$
$e^{-n/\sqrt{2}\,x}\left(c_3 \cosh\sqrt{\frac{1-2n^2}{2}} + c_4 \sinh\sqrt{\frac{1-2n^2}{2}}\right)$. Ans.

Example: $y'' - 4y' - 5y = 0$, $y(0)=1$
$y'(0)=2$.

$\therefore$ The auxiliary equation is given as;

$$m^2 - 4m - 5 = 0$$

$$\therefore \quad m = \frac{+4 \pm \sqrt{16-4(-5)}}{2}$$

$$= \frac{4 \pm \sqrt{16+20}}{2}$$

$$= \frac{4 \pm 6}{2}$$

$$m = 5, -1$$

$$\therefore \quad C.F = y(x) = c_1 e^{5x} + c_2 e^{-1x}$$

$\therefore$

$$y(0) = 1$$

$$1 = c_1 e^{5(0)} + c_2 e^{-1(0)}$$

$$1 = c_1 + c_2 \quad \text{——} \quad (A)$$

$$y'(x) = 5c_1 e^{5x} - c_2 e^{-x}$$

$$y'(0) = 2$$

$$2 = 5c_1 - c_2 \quad \text{——} \quad (B)$$

Adding (A) and (B)

$$\therefore \quad 3 = 6c_1$$

$$\boxed{c_1 = 1/2}$$

and $\boxed{c_2 = 1/2}$

Ans.

Example: $y'' + 4y' + 4y = 0$

$\therefore$ The auxiliary equation is given as;

$m^2 + 4m + 4 = 0$

$m^2 + 2m + 2m + 4 = 0$

$m(m+2) + 2(m+2) = 0$

$m = -2, -2$

$\therefore$ $C.F = (c_1 + xc_2)e^{-2x}$. Ans.

Example: $(D^2 - 2D + 4)^2 y = 0$

$\therefore$ $(D^2 - 2D + 4)(D^2 - 2D + 4)y = 0$

The auxiliary equation is given as;

$m^2 - 2m + 4 = 0$, $m^2 - 2m + 4 = 0$

$m = \dfrac{2 \pm \sqrt{4-16}}{2}$ $m = \dfrac{2 \pm \sqrt{4-16}}{2}$

$m = \dfrac{2 \pm \sqrt{12}i}{2}$ $m = \dfrac{2 \pm \sqrt{12}i}{2}$

$m = 1 \pm i\sqrt{3}$ $m = 1 \pm i\sqrt{3}$.

As, the roots are imaginary but are also repeated.
Hence

$C.F = e^{x}\left((c_1 + xc_2)\cos\sqrt{3}x + (c_3 + xc_4)\sin\sqrt{3}x\right)$ Ans.

Example: Solve D.E

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

$\therefore$ The auxiliary equation is given as;

$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

$\therefore$ $(m^2 - 2m + 2)^2 = 0$

$m = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

As, the roots are imaginary and repeated
Hence,

$C.F = e^{x}\left((c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x\right)$. Ans.

→ Operator

$$L(y) = a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x)$$

$$L(y) = a_0 D^2 + a_1 D + a_2 y = r(x)$$

$$L(y) = [F(D)] = r(x)$$

$$L(y) = [F(D)]^{-1} \cdot r(x) \longrightarrow \text{Functional derivative.}$$

Functional Integral.

∴ Integral Solution: Complementary Function + Particular Integral.

Case 1: If $r(x) = e^{ax}$

$$P.I = \frac{1}{F(D)} \cdot e^{ax}$$

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$$P.I = \frac{1}{F(a)} \cdot e^{ax}$$

Example: $(D+1)^3 y = e^{-x}$

∴ The auxiliary equation is given as;

$$(m+1)^3 = 0$$

$$\Rightarrow \quad m = -1, -1, -1$$

∴ Complementary Function $= (c_1 + x c_2 + x^2 c_3) e^{-x}$.

$$P.I = \frac{1}{F(D)} \cdot e^{ax}$$

$$= \frac{1}{(D+1)^3} \cdot e^{-x}$$

But, according to the P.I of exponential function.

$$P.I = \frac{1}{(D+1)^3} \cdot e^{ax} = \frac{1}{F(a)} \cdot e^{ax}$$

But, as $(D-1)^3 = 0$ for $D = 1$

$\therefore$ Differentiating w.r.t D and multiplying w x

$$P.I = \frac{x}{3(D-1)^2} . e^{-x}$$

Again, differentiating w.r.t D and multiplying with x.

$$P.I = \frac{x^2}{6(D-1)} . e^{-x}.$$

Again, differentiating w.r.t D and multiplying with x

$$P.I = \frac{x^3}{6} . e^{-x}.$$

Hence,

General Solution : $C.F + P.I$

$$y(x) = (C_1 + xC_2 + x^2C_3).e^{-x} + \frac{x^3}{6}.e^{-x}. \text{ Ans.}$$

Example : Solve

$$(D-2)^2 y = 17.e^{2x}$$

$\therefore$ The auxiliary equation is given as;

$$(m-2)^2 = 0$$

$\Rightarrow$ $m = 2, 2$ As, the roots are real and repeated.

Hence, $C.F = (C_1 + xC_2).e^{2x}$

and $P.I = \frac{1}{F(D)} . r(x)$

$$= \frac{1}{(D-2)^2} . 17.e^{2x} = 17 . \frac{1}{(D-2)^2} . e^{2x}$$

As, $(D-2)^2 = 0$ for $a = 2$

$\therefore$ Differentiating w.r.t D and multiplying by x.

$$P.I = \frac{17.x.e^{2x}}{2(D-2)}$$

Again, differentiating w.r.t D and multiplying by x.

$$P.I = \frac{17.x^2.e^{2x}}{2}$$

General Solution: $C.F + P.I$

$$y(x) = (C_1 + xC_2)e^{2x} + \frac{17 \cdot x^2 \cdot e^{2x}}{2}. \quad \underline{\underline{Ans.}}$$

Example: $y'' - 2y' - 3y = 3e^{2x}$.

The auxiliary equation is given as;

$$m^2 - 2m - 3 = 0$$

$$\therefore \quad m^2 - 3m + m - 3 = 0$$

$$m(m-3) + 1(m-3) = 0$$

$$m = -1, 3$$

As, the roots of equation are real and distinct;

Hence,

$$C.F = C_1 e^{-x} + C_2 e^{3x}$$

and

$$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{(D^2 - 2D - 3)} \cdot 3e^{2x}$$

$$P.I = 3 \cdot \frac{1}{D^2 - 2D - 3} \cdot e^{2x}$$

As,

$$P.I = \frac{1}{F(D)} \cdot e^{ax}$$

$$\Rightarrow P.I = \frac{1}{F(a)} \cdot e^{ax}$$

Hence

$$P.I = 3 \cdot \frac{e^{2x}}{4 - 4 - 3} = -e^{2x}$$

$\therefore$ The general solution: $C_1 e^{-x} + C_2 e^{3x} - e^{2x}$. $\underline{\underline{Ans.}}$

Example: $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$

The auxiliary equation is given as;

$$m^3 - 2m^2 - 5m + 6 = 0$$

$$(m-1)(m^2 - m - 6) = 0$$

$$(m-1)(m^2 - 3m + 2m - 6) = 0$$

$$(m-1)(m(m-3) + 2(m-3)) = 0$$

$$(m-1)((m+2)(m-3)) = 0$$

$$\therefore \quad m = 1, 3, -2$$

$1 - 2 - 5 + 6$

$$\begin{array}{r|l} & m^2 - m - 6 \\ m - 1 & m^3 - 2m^2 - 5m + 6 \\ & m^3 - m^2 \\ & - \quad + \\ \hline & -m^2 - 5m \\ & -m^2 + m \\ & + \quad - \\ \hline & -6m + 6 \end{array}$$

As, the roots are real and distinct;

Hence,

$$C.F = c_1 e^{x} + c_2 e^{3x} + c_2 e^{-2x}.$$

and $P.I = \frac{1}{F(D)} . r(x)$

$$= \frac{1}{D^3 - 2D^2 - 5D + 6} . (4e^{-x} - e^{2x})$$

$$= 4 . \frac{e^{-x}}{D^3 - 2D^2 - 5D + 6} - \frac{e^{2x}}{D^3 - 2D^2 - 5D + 6}$$ Replacing $D \to a$

$$= \frac{4 . e^{-x}}{-1 - 2 + 5 + 6} - \frac{e^{2x}}{8 - 8 - 10 + 6}$$

$$= \frac{4 . e^{-x}}{8} - \frac{e^{2x}}{(-4)}$$

$$P.I = \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$ Ans

$\therefore$ The general solution is

$$G.S : C.F + P.I$$

$$: c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$ Ans.

Case 2: If $r(x) = \cos ax$ or $\sin ax$

$$P.I = \frac{1}{F(D)} . r(x)$$ $D^2 \to -a^2.$

$$= [F(D)]^{-1} . \cos ax$$

$$P.I = \frac{1}{F(-a^2)} . \cos ax = \frac{1}{F(-a^2)} . \sin ax$$

→ Solve:

$$y'' + 4y = 6\cos 2x$$

$\therefore$ The auxiliary equation is given as;

$$m^2 + 4 = 0$$

$$\therefore \quad m = \pm 2i$$

As, the roots are imaginary, hence

$$C.F = e^{0x}(c_1 \cos 2x + c_2 \sin 2x).$$ Ans

$$P.I = \frac{1}{F(D)} . r(x)$$

$$= \frac{1}{D^2+4} . 6 . \cos 2x$$

As, $D^2+4=0$, while putting $D^2=-4$

$\therefore$ Differentiating P.I w.r.t D and multiplying by D.

$$P.I = \frac{1}{2D} . x . 6 . \cos 2x$$

$$= \frac{x}{2} . 6 . \left(\frac{1}{D}\right) \cos 2x$$

$$= 3x . \int \cos 2x \, dx$$

$$= 3x . \frac{\sin 2x}{2}$$

$$P.I = \frac{3}{2} x \sin 2x$$

$\therefore$ The general solution:

$$G.S: c_1 \cos 2x + c_2 \sin 2x + \frac{3}{2} x \sin 2x .$$ Ans.

Example: $4y'' - 4y' + y = \sin 3x$

$\therefore$ The auxiliary equation is given as

$$4m^2 - 4m + 1 = 0$$

$$4m^2 - 2m - 2m + 1 = 0$$

$$2m(2m-1) - 1(2m-1) = 0$$

$$\therefore \quad m = 1/2, 1/2$$

As, the roots are real and repeated, hence

$$C.F = (c_1 + x c_2) . e^{1/2 x}$$

and $$P.I = \frac{1}{F(D)} . r(x)$$

$$= \frac{1}{4D^2 - 4D + 1} . \sin 3x$$

$$= \frac{1}{-36 - 4D + 1} . \sin 3x$$

$$= \frac{-1}{4D + 35} . \sin 3x$$

$$P.I = -\frac{1}{4} \cdot \frac{1}{\left(D+\frac{35}{4}\right)} \cdot \sin 3x$$

$$= -\frac{1}{4} \times \frac{\left(D-\frac{35}{4}\right)}{D^2 - \left(\frac{35}{4}\right)^2} \cdot \sin 3x$$

$$= -\frac{1}{4} \times \frac{\left(D-\frac{35}{4}\right)}{-9 - \left(\frac{35}{4}\right)^2} \cdot \sin 3x$$

$$= -\frac{1}{\cancel{4}} \times \frac{\left(D-\frac{35}{4}\right)}{\frac{(-9\times16)-(35)^2}{\cancel{4}}}$$

$$= \frac{D - \frac{35}{4}}{144+(35)^2} \cdot \sin 3x$$

$$P.I = \frac{3\cos 3x - \frac{35}{4}\sin 3x}{1369}$$

Hence, the general solution;

$$G.S = C.F + P.I$$

$$= (C_1 + C_2 x)e^{x/2} + \frac{3}{1369}\cos 3x - \frac{35}{5476} \cdot \sin 3x. \quad \underline{\underline{Ans}}.$$

Q. Solve

$$(D^2+9)y = 6\sin 3x$$

$\therefore$ The auxiliary equation is given as;

$$m^2 + 9 = 0$$

$$\therefore m = 0 \pm 3i = \alpha \pm i\beta$$

Hence, as they roots are imaginary:

$$C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$= e^{0x}(c_1\cos 3x + c_2 \sin 3x)$$

$$\Rightarrow C.F = c_1 \cos 3x + c_2 \sin 3x.$$

and $P.I = \frac{1}{F(D)} \cdot x(x).$

$\therefore \quad P.I = \frac{1}{D^2+9} \cdot 6\sin 3x$

On replacing $D^2 \to -(3)^2$, we get $D^2+9=0$

$\therefore$ Differentiating w.r.t D and multiplying by x

$\Rightarrow \quad P.I = \frac{x}{2D} \cdot 6 \cdot \sin 3x$

$= 3x \cdot \left(\frac{1}{D}\right) \cdot \sin 3x$

$= 3x \int \sin 3x \cdot dx$

$P.I = 3x \times \frac{(-\cos 3x)}{3} = -x\cos 3x$

$\therefore$ The general solution is given as;

$G.S = P.I + C.F$

$= c_1\cos 3x + c_2 \sin 3x - x\cos 3x$

$G.S = (c_1 - x)\cos 3x + c_2 \sin 3x$ Ans.

Q. Solve

$y'' - 3y' - 3y = (-2)\cos 3x$

$\therefore$ The auxiliary equation is given as;

$m^2 - 3m - 3 = 0$

$\therefore \quad m = \frac{3 \pm \sqrt{9+12}}{2}$

$m = \frac{3 \pm \sqrt{21}}{2} = \alpha \pm \sqrt{\beta}$

As, the roots are irrational

$C.F = e^{\alpha x}(c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta})$

$= e^{3/2 x}\left(c_1 \cosh\sqrt{\frac{21}{4}} + c_2 \sinh\sqrt{\frac{21}{4}}\right)$

and $PI = \frac{1}{F(D)} \cdot r(x)$

$= \frac{1}{D^2 - 3D - 3} \cdot (-2)\cos 3x$

$\therefore$ Putting $D^2 \to -9$

$$\begin{array}{r} 144 \\ 81 \\ \hline 225 \end{array}$$

$$\therefore \quad P.I = (+2) \times \frac{1}{+12+3D} \cdot \cos 3x$$

$$= (2) \times \frac{(3D-12)}{9D^2-144} \cdot \cos 3x$$

$$= (2) \times \frac{(3D-12)}{-81-144} \cdot \cos 3x$$

$$= \left(\frac{2}{-225}\right)\left(3D\cos 3x - 12\cos 3x\right)$$

$$= \left(\frac{-2}{225}\right)\left(9(-\sin 3x) - 12\cos 3x\right)$$

$$\Rightarrow P.I = \frac{18}{225}\sin 3x + \frac{24}{225}\cos 3x$$

$\therefore$ The general solution is given as;

$$G.S = P.I + C.F$$

$$y = G.S = e^{3/2 x}\left(c_1 \cosh\sqrt{\frac{21}{4}} + c_2 \sinh\sqrt{\frac{21}{4}}\right) + \frac{18}{225}\sin 3x + \frac{24}{225}\cos 3x$$

Ans.

Q. $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$

$\therefore$ The auxiliary equation will be given as;

$$m^2 - 4m + 1 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$m = 2 \pm \sqrt{3} = \alpha \pm \sqrt{\beta}$$

As, the roots are irrational,

$$C.F = e^{\alpha x}\left(c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta}\right)$$

$$C.F = e^{2x}\left(c_1 \cosh\sqrt{3} + c_2 \sinh\sqrt{3}\right)$$

$$\therefore \quad P.I = \frac{1}{F(D)} \cdot R(x)$$

$$= \left(\frac{1}{D^2-4D+1}\right)\left(\cos x \cos 2x + \sin^2 x\right)$$

$$P.I = \left(\frac{1}{D^2-4D+1}\right)(\cos 2x \cos x) + \left(\frac{1}{D^2-4D+1}\right)(\sin^2 x)$$

$$= \left(\frac{1}{D^2-4D+1}\right)(\cos 3x + \cos x) + \left(\frac{1}{D^2-4D+1}\right)\left(\frac{1}{2} - \frac{\cos 2x}{2}\right)$$

$$= \left(\frac{\cos 3x}{-8-4D}\right) + \left(\frac{\cos x}{-4D}\right) + \left(\frac{e^0}{1}\right)\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{(\cos 2x)}{-3-4D}\right)$$

$$= -\frac{(D-2)\cos 3x}{4(D^2-4)} + \left(-\frac{1}{4}\right)\sin x + \frac{1}{2} + \frac{1}{8}\left(\frac{D-3/4}{D^2-3/4}\right)\cos 2x$$

$$= \left(\frac{D\cos 3x - 2\cos 3x}{4 \times +13}\right) - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{D\cos 2x - 3/4\cos 2x}{-39/4}\right)$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{+2\sin 2x + \frac{3}{4}\cos 2x}{+39/4}\right)$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{8\sin 2x + 3\cos 2x}{39}\right)$$

$\therefore$ General solution is given as;

$$GS = PI + CF$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{8\sin 2x}{39} + \frac{3\cos 2x}{39}\right) +$$

$$e^{2x}(c_1 \cosh\sqrt{3} + c_2 \cosh\sqrt{3})$$ Ans.

Case 3: If $h(x) = x^n$

$$P.I = \frac{1}{F(D)} \cdot x^n$$

$$P.I = [F(D)]^{-1} \cdot x^n$$

$\hookrightarrow$ Binomial expansion

Q. Solve

$$y'' + 16y = 64x^2$$

The auxiliary equation is given as;

$$m^2 + 16 = 0$$

$$\therefore \quad m = \pm 4i = \alpha \pm \beta i$$

As, the roots are imaginary.

Hence;

$$C.F = c_1 \cos 4x + c_2 \sin 4x$$

$$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{D^2+16} \cdot 64x^2$$

$$= \frac{1}{16} \cdot 64 \cdot \left[1 + \left(\frac{D}{4}\right)^2\right]^{-1} \cdot x^2$$

$$= 4 \cdot \left[1 + \frac{D}{4} + \frac{D^2}{16} - \cdots\right] x^2$$

$$= 4\left[x^2 + \frac{2x}{4} + \frac{2}{16}\right]$$

$$= 4x^2 + 2x + \frac{1}{2} \text{ Ans.}$$

Q. Solve

$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$$

$$\therefore \quad m^4 + 3m^2 = 0$$

$$m^2(m^2+3) = 0$$

$$\therefore \quad m = 0, 0, \pm\sqrt{3}i$$

Hence, the complementary function is given as;

$$C.F = (c_1 + xc_2) + (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$$

and $P.I = \frac{1}{F(D)} \cdot r(x)$

$$= \frac{1}{D^4+3D^2} \cdot 108x^2$$

$$= \frac{1}{D^2(D^2 \quad)} \cdot 108x^2 = \frac{1}{3}\left[\frac{D^2+3-D^2}{D^2(D^2+3)}\right] \cdot 108x^2$$

$$\cancel{= \frac{1}{\left(1+\left(\frac{D}{\sqrt{3}}\right)^2\right)} \cdot 108x^2} = \frac{1}{3}\left[\frac{1}{D^2} - \frac{1}{D^2+3}\right] \cdot 108x^2$$

$$= \frac{1}{3}\left[\frac{1}{D^2}\cdot 108x^2 - \frac{1}{D^2+3}\cdot 108x^2\right]$$

$$= \frac{1}{3}\left[108\iint x^2\,dx\,dx - \frac{1}{3\left(1+\left(\frac{D}{\sqrt{3}}\right)^2\right)}\cdot 108x^2\right]$$

$$= \frac{1}{3}\left[\frac{108}{12}x^4 - \frac{1}{3}\left[1+\left(\frac{D}{\sqrt{3}}\right)^2\right]\cdot 108x^2\right]$$

$$= 3x^4 - \frac{1}{9}\left[1 - \frac{D^2}{3}\right]108x^2$$

$$= 3x^4 - \frac{1}{9}\left[108x^2 - \frac{2\times 108}{3}\right]$$

$$= 3x^4 - 12x^2 + \frac{2\times 108}{3\times 9}$$

$$P.I = 3x^4 - 12x^2 + 8$$

Hence, the general solution is given as;

$$GS = P.I + CF$$

$$= (c_1 + xc_2) + (c_3\cos\sqrt{3}x + c_4\sin\sqrt{3}x) + 3x^4 - 12x^2 + 8$$

Ans.

Q. Solve D.E

$$(D^3 - 7D^2 + 6)y = 1 + x^2$$

∴ The auxiliary equation can be given as;

$$D^3 - 7D^2 + 6 = 0$$

$$\therefore \quad m^3 - 7m^2 + 6 = 0$$

$$\Rightarrow \quad (m-1)(m^2 - 6m - 6) = 0$$

$$\therefore \quad m = 1,\ \frac{6 \pm \sqrt{36+24}}{2}$$

$$m = 1,\ 3 \pm \sqrt{15}$$

$$\begin{array}{r|l} & m^2 - 6m - 6 \\ \hline m-1 & m^3 - 7m^2 + 6 \\ & m^3 - m^2 \\ & \ominus \quad \oplus \\ \hline & -6m^2 + 6 \\ & -6m^2 + 6m \\ & + \quad + \\ \hline & -6m + 6 \\ & -6m + 6 \\ & \oplus \quad \ominus \end{array}$$

Hence, the complementary function CF can be given as;

$$CF = c_1e^x + e^{3x}(c_2\cosh\sqrt{15} + c_3\sinh\sqrt{15}).$$

and $P.I = \frac{1}{F(D)}\cdot r(x)$

$$P.I = \frac{1}{(D^3-7D^2+6)} . (1+x^2)$$

$$= \left(\frac{1}{D^3-7D^2+6} . e^{0x}\right) - \left(\frac{1}{D^3-7D^2+6}\right) x^2$$

$$= \frac{1}{6} - \frac{1}{6}\left[1+\frac{D^3-7D^2}{6}\right]^{-1} . x^2$$

$$(1+ax)^{-1} = 1 - ax + \frac{a(a+1)}{2!}x^2 - \frac{a(a+1)(a+2)}{3!}x^3 + \dots$$

$$\therefore PI = \frac{1}{6} - \frac{1}{6}\left[1 - \frac{D^3-7D^2}{6}\right] . x^2$$

$$= \frac{1}{6} - \frac{1}{6}\left[x^2 - 0 + \frac{7 \times 2}{6}\right]$$

$$P.I = \frac{1-x^2-14}{6}$$

$\therefore$ The general solution is given by;

$$GS = CF + PI$$

$$GS = c_1 e^x + e^{3x}(c_2 \cosh\sqrt{15} + c_3 \cosh\sqrt{15}) + \frac{1-x^2-14}{6}$$

Ans.

Q. Solve:

$$(D^3 - D^2 - 6D)y = 1 + x^3$$

$\therefore$ The auxiliary equation is given as;

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m^2 - 3m + 2m - 6) = 0$$

$$m(m(m-3) + 2(m-3)) = 0$$

$$m(m+2)(m-3) = 0$$

$$\Rightarrow m = 0, -2, 3$$

As, the roots are real and distinct;

$$\Rightarrow CF = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$\therefore PI = \frac{1}{F(D)} . r(x) = \frac{1}{(D^3-D^2-6D)} . (1+x^3)$$

$$= \left[\frac{1}{D^3-D^2-6D}\right] + \frac{1}{(-6D)} . \left[1 - \left(\frac{D^2-D}{6}\right)\right]^{-1} . x^3$$

$$= \left[\frac{x}{3D^2-2D-6} . e^{0x}\right] - \frac{1}{6D}\left[1 + \frac{D^2-D}{6} + \frac{D^4+D^2-2D^3}{36}\right] . x^3$$

$$= \left(\frac{x}{-6}\right) - \frac{1}{6D}\left[x^3 + \frac{6x}{6} - \frac{3x^2}{6} + 0 + \frac{6x}{36} - \frac{12}{36}\right]$$

$$\frac{x-3x}{18}$$

$$P.I = -\frac{x}{6} - \frac{x^4}{24} - \frac{x^2}{12} + \frac{x^3}{36} - \frac{x^2}{72} + \frac{1}{18}x$$

$$P.I = -\frac{x}{9} - \frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72}$$

Hence, the general solution is given as;

$$GS = PI + CF$$

$$= c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72} - \frac{x}{9}. \text{ Ans.}$$

→ Case 4: If V is any function

$$P.I = \frac{1}{F(D)} \cdot e^{ax} \cdot V$$

$$P.I = e^{ax} \cdot \frac{1}{F(D+a)} \cdot V$$

Q. Solve D.E

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = (1-x)e^{2x}$$

∴ The auxiliary equation is given as;

$$m^2 - 7m + 12 = 0$$

$$\Rightarrow m^2 - \cancel{7}3m - 4m + 12 = 0$$

$$m(m-3) - 4(m-3) = 0$$

$$(m-3)(m-4) = 0$$

$$\Rightarrow m = 3, 4.$$

∴ As, the roots are real and distinct;

$$C.F = c_1 e^{3x} + c_2 e^{4x}.$$

$$\therefore PI = \left[\frac{1}{D^2 - 7D + 12}\right] \cdot e^{2x}(1-x)$$

$$= e^{2x} \cdot \frac{1}{D^2 + 4 + 4D - 7D - 14 + 12} \cdot (1-x)$$

$$= e^{2x} \cdot \frac{1}{D^2 - 3D + 2} \cdot (1-x)$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[1+\frac{D^2-3D}{2}\right]^{-1}.x\right.$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[1-\frac{D^2+3D}{2}\right].x\right.$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[x - \frac{0}{2} + \frac{3}{2}\right]\right]$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}x - \frac{3}{4}\right]. \text{ Ans.}$$

Hence, General solution is given as;

$$GS = PI + CF$$

$$= c_1 e^{3x} + c_2 e^{4x} + e^{2x}\left[\frac{1}{2} - \frac{x}{2} - \frac{3}{4}\right]. \quad \text{Ans.}$$

Q. Solve D.E

$$y'' - 2y' + 2y = e^{2x}\cos 2x$$

∴ The auxiliary equation can be given as;

$$m^2 - 2m + 2 = 0$$

$$m^2 = \frac{+2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$m = 1 \pm 2i = \alpha \pm i\beta$$

As, the roots are imaginary;

$$C.F = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$= e^{x}(c_1\cos 2x + c_2\sin 2x).$$

$$\therefore \quad P.I = \frac{1}{F(D)}.\,r(x)$$

$$= \frac{1}{D^2-2D+2}.\,e^{2x}\cos 2x$$

$$= e^{2x}.\,\frac{1}{(D+2)^2-2(D+2)+2}.\cos 2x$$

$$= e^{2x}.\,\frac{1}{D^2+4+4D-2D-4+2}.\cos 2x$$

$$= e^{2x}.\,\frac{1}{-4+\cancel{4}+2D-\cancel{4}+2}.\cos 2x = e^{2x}.\,\frac{1}{2D-2}.\cos 2x$$

$$= \frac{e^{2x}}{2} \cdot \frac{D+1}{D^2-1} \cdot \cos 2x$$

$$= \frac{e^{2x}}{2} \cdot \frac{(D\cos 2x + \cos 2x)}{-5}$$

$$= \frac{e^{2x}}{2} \cdot \left(\frac{-2\sin 2x + \cos 2x}{-5}\right). \quad \text{Ans.}$$

$\therefore$ The general solution is given as;

$$GS = CF + PI$$

$$= e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{e^{2x}}{10}(2\sin 2x - \cos 2x)$$

Ans.

Q. Solve

$$D^2 - 4D + 5y = e^{x} \cos \frac{x}{2}.$$

$\therefore$ The auxiliary equation is given as;

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16-20}}{2}$$

$$m = \frac{4 \pm 2i}{2} = 2 \pm i = \alpha \pm i\beta$$

$$\therefore \quad C.F = e^{2x}(c_1 \cos x + c_2 \sin x)$$

$$P.I = \frac{1}{F(D)} \cdot h(x)$$

$$= \frac{1}{D^2 - 4D + 5} \cdot e^{x} \cdot \cos \frac{x}{2}$$

$$= e^{x} \cdot \frac{1}{D^2 + 1 + 2D - 4D - 4 + 5} \cdot \cos \frac{x}{2}$$

$$= e^{x} \cdot \frac{1}{-\frac{1}{4} + 2 - 2D} \cdot \cos \frac{x}{2}$$

$$4e^x \cdot \frac{1}{7-8D} \cdot \cos\frac{x}{2}$$ $\frac{49}{16}$ $\overline{5}$

$$\frac{4}{(-8)} \cdot e^x \cdot \frac{1}{D-\frac{7}{8}} \cdot \cos\frac{x}{2}$$

$$-2 \cdot e^x \cdot \frac{D+\frac{7}{8}}{-\frac{1}{4}-\frac{49}{64}} \cdot \cos\frac{x}{2}$$

$$\frac{-2e^x}{\left(\frac{-16-49}{64}\right)} \cdot \left(\frac{-\sin\frac{x}{2}}{2} + \frac{7}{8}\cos\frac{x}{2}\right)$$

$$= \frac{128}{65}e^x \cdot \left[\frac{7}{8}\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right] \cdot \text{Ans}$$

Hence, the general solution is given as;

$$GS = CF + PI$$

$$= e^{2x}(c_1\cos x + c_2 \sin x) + \frac{128}{65}e^x \cdot \left[\frac{7}{8}\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right] \cdot \underline{\underline{\text{Ans}}}.$$

Euler - Cauchy differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = r(x)$$

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = r(x)$$

$$\underbrace{a_0, a_1, a_2 \text{ are constants}}$$

Linear differential equation with constant coefficients.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = r(x)$$

Linear non-homogenous differential equations.

=> Method to solve Euler- Cauchy differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = r(x)$$

$$x\frac{dy}{dx} \longrightarrow \text{constant coefficient}$$

$$x = e^z, \quad z = \ln x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{x} = \frac{1}{x}\frac{dy}{dz}. \quad \Rightarrow \frac{dy}{dz} = \frac{x\,dy}{dx}.$$

$\therefore \quad x\dfrac{dy}{dx} \equiv Dy \quad , \quad D \equiv \dfrac{d}{dz}$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]$

$= \dfrac{d}{dx}\left[\dfrac{1}{x}\dfrac{dy}{dz}\right]$

By using product rule,

$= \dfrac{dy}{dz}\cdot\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x}\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\dfrac{dz}{dx}$

$= \dfrac{dy}{dz}\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x}\dfrac{d^2y}{dx^2}\cdot\dfrac{dz}{dx}$

$= \dfrac{dy}{dz}\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\cdot\dfrac{1}{x}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{1}{x^2}\dfrac{d^2y}{dz^2} - \dfrac{1}{x^2}\dfrac{dy}{dz}$

$x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} = D^2y - Dy.$

Q. $\left(D(D-1)+2D+2\right)y = 10\left(e^z + \dfrac{1}{e^z}\right)$

$\therefore \quad (D^2+D+2)y = 10\left(e^z + \dfrac{1}{e^z}\right)$

The auxiliary equation is given as;

$m^2 + m + 2 = 0$

$\therefore \quad m = \dfrac{-1 \pm \sqrt{1-8}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{7}}{2}i$

Hence,

$C.F = e^{-1/2 z}\left(c_1\cos\dfrac{\sqrt{7}}{2}z + c_2\sin\dfrac{\sqrt{7}}{2}z\right)$

$\therefore \quad P.I = \dfrac{1}{F(D)}\cdot n(x)$

$= \dfrac{1}{D^2+D+2}(e^z+e^{-z}) = \dfrac{1}{4}e^z + \dfrac{1}{2}e^{-z}$

$\Rightarrow G.S = e^{-1/2 z}\left(c_1\cos\dfrac{\sqrt{7}}{2}z + c_2\sin\dfrac{\sqrt{7}}{2}z\right) + \dfrac{1}{4}e^z + \dfrac{1}{2}e^{-z}$ Ans.

Q. $D(D-1)(D-2)y + 3D(D-1)y + Dy + y = e^z + z$

$$(D^3 - \cancel{2D^2} - \cancel{D^2} + \cancel{2D} + \cancel{3D^2} - \cancel{3D} + \cancel{D} + 1)y = e^z + z$$

$$\Rightarrow \quad (D^3+1)y = e^z + z$$

Hence, the auxiliary equation is given as:

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \ \frac{1 \pm i\sqrt{3}}{2}$$

Hence,

$$CF = c_1 e^{-z} + e^{-1/2 z}\left[c_1 \cos\frac{\sqrt{3}}{2}z + c_2 \sin\frac{\sqrt{3}}{2}z\right]$$

$$\Rightarrow \quad P.I = \frac{1}{F(D)} \cdot e^z \cdot z$$

$$= \frac{1}{D^3+1} \cdot (e^z + z)$$

$$= \frac{1}{2}e^z + [1 - D^3]z$$

$$P.I = \frac{e^z}{2} + z$$

Hence;
$$GS = CF + PI$$
$$GS = c_1 e^{-z} + e^{-z/2}\left[c_1 \cos\frac{\sqrt{3}}{2}z + c_2 \sin\frac{\sqrt{3}}{2}z\right] + \frac{e^z}{2} + z. \quad \text{Ans.}$$

→ Method of variation of parameters

Consider the second order differential equation:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x)$$

(a) Complementary function:

$$R = \frac{r(x)}{a_0}$$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore \quad y = A(x) e^{m_1 x} + B(x) e^{m_2 x}$$
$$y = A(x).u + B(x)v$$

$$\therefore \quad A(x) = -\int \frac{Rv}{W} dx + a$$

$$B(x) = +\int \frac{Ru}{w}\,dx + b$$

$W \rightarrow$ Wronskian

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

Q. Solve the D.E

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

$\therefore$ The auxiliary equation is given as;

$$m^2 + a^2 = 0$$
$$m = \pm ai$$

$\therefore$ $C.F = c_1 \cos ax + c_2 \sin ax$

$\Rightarrow$ $y = A(x)\cos ax + B(x)\sin ax$ $\qquad R = \sec ax$

$y = A(x)u + B(x)v$

$\Rightarrow$ $u = \cos ax$, $v = \sin ax$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$\therefore \quad A(x) = -\int \frac{\sec ax \cdot \sin ax}{a} \cdot dx + a'$$

$$= \frac{-1}{a}\int \tan ax\, dx + a'$$

$$A(x) = -\frac{1}{a}\ln|\sec x| + a'$$

and $B(x) = \int \frac{Ru}{w}dx + b' = \int \frac{\sec ax \cdot \cos ax}{a}dx + b'$

$$B(x) = \frac{x}{a} + b'$$

Hence;

$$y = +\left(\frac{1}{a}\ln|\cos x| + a'\right)\cos ax + \left(\frac{x}{a} + b'\right)\sin ax. \quad \text{Ans.}$$

Q. Solve the following differential equations by variation of parameters.

(a) $\frac{d^2y}{dx^2} + y = \tan x$

$\therefore$ The auxiliary equation is given as;

$m^2 + 1 = 0$

$m = \pm i$

$\therefore$ C.F $= c_1 \cos x + c_2 \sin x$

$\Rightarrow y = A(x)\cos x + B(x)\sin x$

$y = A(x)u + B(x)v$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\Rightarrow u = \cos x$ and $v = \sin x$

$\therefore A(x) = -\int \frac{Rv}{W} dx + a$

$= -\int \frac{\tan x \cdot \sin x}{1} \cdot dx + a$

$= -\int \frac{\sin^2 x}{\cos x} \cdot dx$

$= -\int \frac{1 - \cos^2 x}{\cos x} dx$

$= -\int \sec x \, dx + \int \cos x \, dx$

$A(x) = -\ln|\sec x + \tan x| + \sin x + a$

and $B(x) = \int \frac{Ru}{W} dx + b$

$= \int \frac{\tan x \cdot \cos x}{1} dx + b$

$= \int \sin x \, dx + b$

$= -\cos x + b$

$\Rightarrow y = (-\ln|\sec x + \tan x| + \sin x + a)\cos x + (-\cos x + b)\sin x$

Ans.

(b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x$

$\therefore$ The auxiliary equation is given as;

$$m^2 - 2m = 0 \qquad\qquad R = e^x \sin x$$
$$m(m-2) = 0$$

$$\Rightarrow \quad C.F = c_1 + c_2 e^{2x}$$

Hence, $y = A(x).1 + B(x).e^{2x}$

$$y = A(x)u + B(x)v \qquad\qquad W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

$$u = 1, \quad v = e^{2x}$$

$$\therefore \quad A(x) = -\int \frac{Rv}{W}\,dx + a$$

$$= -\int \frac{e^x \sin x . \cancel{e^{2x}}}{2\cancel{e^{2x}}} + a$$

$$A(x) = -\frac{1}{2}\int e^x \sin x\,dx + a$$

Let,

$$I = \int \underset{II}{e^x}\,\underset{I}{\sin x}\,dx = \sin x\, e^x - \int \underset{I}{\cos x}\,\underset{II}{e^x}\,dx$$

$$I = \sin x\, e^x - \cos x\, e^x + \int (-\sin x)e^x\,dx$$

$$\Rightarrow \quad I = \frac{e^x(\sin x - \cos x)}{2}$$

$$\Rightarrow \quad A(x) = \frac{e^x(\cos x - \sin x)}{4} + a$$

and $\quad B(x) = \int \dfrac{Ru}{W}\,dx = \int \dfrac{e^x \sin x}{e^{2x}}.\,dx$

$$B(x) = \int e^{-x} \sin x\,dx$$

$$\therefore \quad I = \int \underset{II}{e^{-x}}\,\underset{I}{\sin x}\,dx$$

$$= -\sin x\, e^{-x} - \int (+\underset{I}{\cos x}).\,(\underset{II}{+e^{-x}}).\,dx$$

$$= -\sin x\, e^{-x} - \left[\cos x(-e^{-x}) - \int \sin x(-e^{-x})\,dx\right]$$

$$\therefore \quad I = \frac{-e^{-x}(\sin x + \cos x)}{2} + b = B(x).$$

$\therefore$ Hence,

$$y = A(x) + B(x)e^{2x}$$

$$= \left(\frac{e^{x}}{4}(\cos x - \sin x) + a\right) + \left(b - \frac{e^{-x}}{2}(\sin x + \cos x)\right)e^{2x}.$$

Ans.

Q. By using the method of variation of parameter solve the following differential equation.

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3\log x$$

$\therefore$ Using Euler-Cauchy differential equation;

$$x = e^{z}$$
$$z = \ln x.$$

$\therefore$ $D(D-1)y + 2Dy - 12y = x^3\log x$

$\Rightarrow$ $D^2y + Dy - 12y = x^3\log x$

$\therefore$ The auxiliary equation is given as;

$$m^2 + m - 12 = 0$$
$$m^2 + 4m - 3m - 12 = 0$$
$$m(m+4) - 3(m+4) = 0$$
$$(m-3)(m+4) = 0$$

$$W = \begin{vmatrix} e^{3z} & e^{-4z} \\ 3e^{3z} & -4e^{-4z} \end{vmatrix}$$

$$= -4e^{-z} - 3e^{-z}$$

$$W = -7e^{-z}$$

$$C.F = c_1e^{3z} + c_2e^{-4z}$$

$$R = x\log x$$
$$= e^{z}.z$$

$$y = A(z)e^{3z} + B(z)e^{-4z}$$

$$y = A(z)u + B(z)v$$

$\Rightarrow$ $u = e^{3z}$, $v = e^{-4z}$

$$x = e^{z}$$
$$dx = e^{z}dz$$

$\Rightarrow$ $A(z) = -\int \frac{Rv}{W}dx + a$

$$A(z) = +\int \frac{e^{z}z.e^{-4z}}{+7e^{-z}}dx + a$$

$$A(z) = \frac{1}{7}\int e^{2z-4z+z}z.dz + a$$

$$A(z) = \frac{1}{7}\int \underset{II}{e^{-z}}.\underset{I}{z}.dz + a$$

$$A(z) = \frac{1}{7}\left[-z e^{-z} + \int e^{-z} dz\right] + a$$

$$A(z) = \frac{1}{7}\left[-z e^{-z} - e^{-z}\right] = -\frac{e^{-z}}{7}\left[z+1\right] + a$$

and

$$B(z) = \int \frac{Ru}{w} dx + b$$

$$= \int \frac{e^{z} \cdot z \cdot e^{3z}}{-7 e^{-z}} \cdot e^{z} dz + b$$

$$= \int \frac{e^{z+3z+z+z}}{-7} \cdot z \, dz + b$$

$$= \frac{-1}{7} \int \underset{II}{e^{6z}} \cdot \underset{I}{z} \, dz + b$$

$$\therefore B(z) = -\frac{1}{7}\left[\frac{z e^{6z}}{6} - \int \frac{e^{6z}}{6} dz\right] + b$$

$$= -\frac{1}{7}\left[\frac{z e^{6z}}{6} - \frac{e^{6z}}{36}\right] + b$$

$$\therefore y(z) = \left(-\frac{e^{-z}}{7}\left[z+1\right] + a\right) e^{3z} + \left(\frac{e^{6z}}{42}\left[\frac{1}{6} - z\right] + b\right) e^{-4z}$$

$$y(x) = \left(-\frac{e^{-\ln x}}{7}\left[\ln x + 1\right] + a\right) e^{3\ln x} + \left(\frac{e^{6\ln x}}{42}\left[\frac{1}{6} - \ln x\right] + b\right) e^{-4\ln x}$$

$$y(x) = \left(\frac{-1}{7x}\left[\ln x + 1\right] + a\right) x^{3} + \left(\frac{x^{6}}{42}\left[\frac{1}{6} - \ln x\right] + b\right) \frac{1}{x^{4}} \quad \text{Ans.}$$

→ To find the complete solution of the differential equation.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

by the method of change of independent variable.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} + \frac{dy}{dz} \cdot \frac{d^2z}{dx^2}$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{dy}{dz}\right)\left(\frac{d^2z}{dx^2}\right)$$

$\therefore$ Subsituting $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in given differential equation;

We get

$$\frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \left[\frac{Q}{\left(\frac{dz}{dx}\right)^2}\right]y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \quad \text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Differential equation obtained after changing the independent variable from 'x' to 'z':

$$\therefore \quad \frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

Q. Solve the D.E:

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 2x^2 y = x^4$$

Hence,

$$P = -1/x$$

$$Q = 2x^2$$

and $R = x^4$

$$\Rightarrow \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

Let $\boxed{Q_1 = 1}$;

$$\left(\frac{dz}{dx}\right)^2 = 2x^2$$

$$\boxed{\frac{dz}{dx} = \sqrt{2}x}$$

$$\therefore \quad \int dz = \int \sqrt{2}x \, dx$$

$$\boxed{z = \frac{x^2}{\sqrt{2}}},$$

and

$$\boxed{\frac{d^2z}{dx^2} = \sqrt{2}}$$

Hence, we get

$$P_1 = \left[\frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)}\right] = \frac{\sqrt{2} - \frac{1}{x}\cdot\sqrt{2}x}{(\sqrt{2}x)^2} = 0$$

$$\therefore \quad R_1 = \frac{x^4}{(\sqrt{2}x)^2} = \frac{x^2}{2}$$

$$D \equiv \frac{d}{dz}$$

Hence, we get

$$D^2y + 0Dy + y = \frac{x^2}{2}.$$

$\therefore$ The auxiliary equation is given as:

$$m^2 + 1 = 0$$

$$\Rightarrow \quad m = \pm i$$

Hence $CF = C_1 \cos z + C_2 \sin z = C_1 \cos(\frac{\sqrt{2}x^2}{2}) + C_2 \sin \frac{x^2}{\sqrt{2}}$

and $PI = \frac{1}{F(D)} \cdot r(z)$

$$= \frac{1}{D^2+1} \cdot \frac{z}{\sqrt{2}}$$

$$= [1 + D^2]^{-1} \cdot \frac{z}{\sqrt{2}}$$

$$= [1 - D^2] \frac{z}{\sqrt{2}}$$

$$P.I = \frac{z}{\sqrt{2}} = \frac{x^2}{2}.$$

$\therefore$ The General solution is given as;

$$GS = C.F + P.I$$

$$G.S = C_1 \cos\left(\frac{x^2}{\sqrt{2}}\right) + C_2 \cos\left(\frac{x^2}{\sqrt{2}}\right) + \frac{x^2}{2} \quad \text{Ans.}$$

Q. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = \frac{\cos(\log(1+x))}{(1+x)^2}$

$$\therefore \quad \frac{d^2y}{dx^2} + \left(\frac{1}{1+x}\right) \frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{\cos(\log(1+x))}{(1+x)^4}$$

$$\therefore \quad P = \frac{1}{1+x}$$

$$Q = \frac{1}{(1+x)^2}$$

$$R = \frac{\cos(\log(1+x))}{(1+x)^4}$$

$\therefore \quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$\therefore \quad \left(\dfrac{dz}{dx}\right)^2 \times Q_1 = \dfrac{1}{(1+x)^2}$

Let $\boxed{Q_1 = 1}$

$\dfrac{dz}{dx} = \dfrac{1}{1+x}$ and $\dfrac{d^2z}{dx^2} = \dfrac{-1}{(1+x)^2}$

$\therefore \quad \boxed{z = \ln(1+x)}$

$\therefore \quad P_1 = \dfrac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = \dfrac{\frac{-1}{(1+x)^2} + \left(\frac{-1}{1+x}\right)\left(\frac{1}{1+x}\right)}{\left(\frac{1}{1+x}\right)^2} = 0$

$\Rightarrow \quad \boxed{P_1 = 0}$

and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2} = \dfrac{\cos(\log(1+x))}{(1+x)^4 \cdot \left(\frac{1}{1+x}\right)^2} = \dfrac{\cos(\log(1+x))}{(1+x)^2}$

$\therefore \quad D \equiv \dfrac{d}{dz}$

D.E obtained:

$$D^2y + 0\,Dy + 1.y = \frac{\cos z}{e^{2z}}$$

$\therefore$ The auxiliary equation is given as:

$m^2 + 1 = 0$

$m = \pm i$

Hence, $CF = c_1 \cos z + c_2 \sin z$

$CF = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+z))$

Now;

$PI = \dfrac{1}{F(D)} \cdot r(z)$

$= \dfrac{1}{(D^2+1)} \cdot e^{-2z} \cos z$

$= e^{-2z} \cdot \dfrac{1}{(D-2)^2+1} \cdot \cos z$

$= e^{-2z} \cdot \dfrac{1}{D^2+4-4D+1} \cdot \cos z$

$= e^{-2z} \cdot \dfrac{1}{4-4D} \cdot \cos z$

$= \dfrac{e^{-2z}}{-4} \times \dfrac{(D-1)}{(-2)} \cdot \cos z$

$= \dfrac{e^{-2z}}{} (-\sin z - \cos z)$

$\therefore PI = -\frac{(1+x)^{-2}}{8}\left(\sin(\ln(1+x)) + \cos(\ln(1+x))\right)$

$\therefore$ General solution is given as;

$GS = CF + PI$

$y = GS = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) - \frac{(1+x)^{-2}}{8}$

$(\sin(\ln(1+x)) + \cos(\ln(1+x)))$

Ans.

These handwritten notes are of MTH-S102 taught to us by Prof. D.K. Singh, compiled and organized chapter-wise to help our juniors. We hope they make your prep a bit easier.

— **Saksham Nigam** and **Misbahul Hasan** (B.Tech. CSE(2024-28))