Eigen Values

Proper values, latent values, characteristic values spectral values.

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n over the field F. An element λ in F is called an Eigen value of A if $|A - \lambda I| = 0$ where I is the unit matrix of order n.

Procedure

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

characteristic matrix

$$[A - \lambda I] = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic Polynomial

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

Characteristic Equation :

$$|A - \lambda I| = 0$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

characteristic values.

$$\lambda = 1, 1, 5$$

Properties of Eigen values

- Any square matrix A and its transpose A^T will have the same eigen values.
- For symmetric matrix eigen values are always real.
- For skew-symmetric matrix, the eigen values are either zero or pure imaginary.
- For triangular matrix (either upper or lower triangular matrix) or diagonal matrix, the eigen values are the main diagonal elements.
- For orthogonal matrix, the mod values of eigen values is always unity. ($1/\lambda$ is also its eigen value)
- If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigen values of A.

(i) the eignen values of KA are

$$K\lambda_1, K\lambda_2, K\lambda_3, \ldots, K\lambda_n$$

(ii) the eigen values of A^m are

$$\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$$

(iii) the eigen values of A^{-1} are

$$1/\lambda_1, 1/\lambda_2, 1/\lambda_3, \ldots, 1/\lambda_n$$

- The eigen values of an idempotent matrix are either zero or unity.

Important points on Eigen values
(shortcuts)

1) The characteristic eqⁿ of the matrix A of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where,

S_1 = sum of principal diagonal elements (trace)

S_2 = determinant A

2) The characteristic eqⁿ. of the matrix A of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where,

S_1 = sum of principal diagonal elements

S_2 = sum of minors of principal diagonal elements

$S_3 = |A|$

3) The sum of eigen values of a matrix is the sum of its principal diagonal elements (or trace of the matrix).

4) The product of eigen values of a matrix is the determinant of the matrix.

Rank = No. of non-zero eigen values

Eigen Vectors

Definition:

Let λ be an eigen value of n square matrix A, then a non-zero matrix X of the order $n \times 1$ (i.e., column matrix) such that $(A - \lambda I) X = 0$.
is called an eigen vector of A corresponding to that eigen value λ.

Procedure to find Eigen vectors

If λ is an eigen value of A then the corresponding eigen vectors of A will be given by a non-zero vector.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

Satisfying the eqn:

$$(A - \lambda I) X = 0$$

$$\Rightarrow \boxed{AX = \lambda X} \longrightarrow \text{Shortcut trick}$$

Use for MCQ's

Properties of Eigen Vectors

1. The eigen vector X of a matrix A is not unique.
2. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of $n \times n$ matrix, then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ forms a linearly independent set.
3. If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to equal matrix.
4. Eigen vectors of a symmetrical matrix corresponding to different eigen values are orthogonal.

Note: Two eigen vectors X_1 and X_2 are called orthogonal vectors if

$$X_1^T \cdot X_2 = 0$$
$$\text{or } X_1 \cdot X_2^T = 0$$

☆ Important

1) Sum of eigen values = Trace of a matrix
2) Product of eigen values = Determinant of a matrix
3) For a real matrix, if $\alpha + i\beta$ is an eigen value, $\alpha - i\beta$ will also be an eigen value

Normalised form of vectors

To find the normalised form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$,
we divide each element by
$\sqrt{a^2+b^2+c^2}$

Example: Normalised form of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$\sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic eqn

Let A be a square matrix and
$\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0$
be its characteristic eqn. then according to Cayley - Hamilton theorem,
$A^3 - 2A^2 + 3A - 4I = 0$

To find A^{-1}

$A^2 - 2A + 3I - 4A^{-1} = 0$
$4A^{-1} = A^2 - 2A + 3I$
$A^{-1} = \frac{1}{4}\left[A^2 - 2A + 3I\right]$

Diagonalisation of a Matrix :

It is a process of reduction of A to a diagonal form D.
If A is related to D by a similarity transformation such that

$$P^{-1} \cdot A \cdot P = D$$

then A is reduced to the diagonal matrix D through modal matrix P.

Important points

1. The matrix P which diagonalises A is called modal matrix of A.
2. Modal matrix is formed by grouping the eigen vectors of A into square matrix
3. The resulting diagonal matrix D is called spectral matrix of A.
4. Spectral matrix has the eigen values of A as its diagonal elements.
5. The transformation of matrix A to $P^{-1} \cdot A \cdot P$ is called "SIMILARITY TRANSFORMATION"

(repeated eigenvalues)
Remember: If two eigen values are same, we make the second and third row zero, (for $n=3$) and then put $x_1=0$ and find X_1 and then $x_2=0$ and find X_2. Hence, we find two set of eigen vectors.

☆ In other case, we use Cramer's rule: $x_1 = -x_2 = x_3$