Differential Equation

→ Homogenous Equation

$$m^2 + 6m + 6 = 0$$

→ Non-Homogenous Equation

$$m^2 + 5m + 6 = e^x.$$

Constant and Arbitrary constant

$$\frac{d}{dx} c = 0$$

Operator: To change the nature of the function.

↳ Rate of change of 'c' w.r.t x but 'c' is constant.

Constant: A function whose rate of change can be calculated is known as a variable, a function whose rate of change cannot be calculated is known as constant.

→ Arbitrary constant:

Role of arbitrary constant to decide order of differential equation.

Integration is the inverse of differentiation.

Rate of change = work done

Arbitrary constant => Family of curve

$f(x, y, c)$, c is the arbitrary constant.

→ Formation of the differential equation

Example: Construct the differential equation for the function $y(x) = c_1 e^x + c_2 e^{-x}$, where c_1 and c_2 are arbitrary constants.

Given,

$$y(x) = c_1 e^x + c_2 e^{-x}$$

$$\frac{dy}{dx}(x) = c_1 e^x - c_2 e^{-x}$$

$$\frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x}$$

$$\therefore \frac{d^2y(x)}{dx^2} = y(x)$$

$$\frac{d^2y(x)}{dx^2} - y(x) = 0.$$

Ans

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Example: Construct the differential equation for the function.

$$y(x) = C_1 \sin x + C_2 \cos x$$

$\therefore$
$$y(x) = C_1 \sin x + C_2 \cos x$$

$$\frac{dy(x)}{dx} = C_1 \cos x - C_2 \sin x$$

$$\frac{d^2y(x)}{dx^2} = -C_1 \sin x - C_2 \cos x = -y$$

$\therefore$
$$\frac{d^2y(x)}{dx^2} + y = 0. \quad \underline{\underline{\text{Ans.}}}$$

Example: Construct the differential equation for the function.

$$y = cx + \frac{1}{c}, \quad c \neq 0$$

$\therefore$
$$y(x) = cx + \frac{1}{c}$$

$\Rightarrow$
$$\frac{dy(x)}{dx} = c$$

$\therefore$
$$y = \left(\frac{dy}{dx}\right)x + \frac{1}{\left(\frac{dy}{dx}\right)}$$

$\Rightarrow$
$$\frac{dy}{dx}y = \left(\frac{d^2y}{dx^2}\right)^2 x + 1$$

$\therefore$
$$x\left(\frac{d^2y}{dx^2}\right)^2 - y\frac{dy}{dx} + 1 = 0$$

Order: 1
Degree: 2

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→ Wronskian Method to determine dependent and independent form of the solution of differential equation

Solution

Explicit (स्पष्ट) (سپشٹ)

→ $y = f(x)$

→ Single independent variable.

Implicit (अस्पष्ट) (اسپشٹ)

→ $z = f(x, y)$

→ More than one independent variable.

$$W(x) = \begin{vmatrix} W_1 & W_2 & W_3 \\ W_1' & W_2' & W_3' \\ W_1'' & W_2'' & W_3'' \end{vmatrix}$$

→ If $W(x) \neq 0$ (Independent)

→ If $W(x) = 0$ (Dependent)

Example: Find the nature of the solution of the given differential equation;

$$y = c_1 e^{x} + c_2 e^{-x}$$

$W_1(x) = e^{x}$, $W_2(x) = e^{-x}$

$$W(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{0} - e^{0} = -2$$

Hence, solution is independent.

Example: Find the nature of the solution of the given differential equation;

$$y = c_1 \sin x + c_2 \cos x$$

$\therefore$ $W_1(x) = \sin x$

$W_2(x) = \cos x$

$$\therefore \quad W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Hence, solution is independent.

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→ Linear differential equation with constant coefficient

Consider the differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

a_0, a_1, a_2

Constant — Linear differential equation with constant variable

variable — Linear non-homogenous differential equation.

→ Auxiliary Equation and Auxiliary Equation roots.

Auxiliary equation:

If we have:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

$\Rightarrow$ Writing

$$\frac{d}{dx} \equiv m$$

$\therefore$ The auxiliary equation is given as:

$$a_0 m^2 y + a_1 m y + a_2 y = R(x)$$

$$\therefore \underbrace{(a_0 m^2 + a_1 m + a_2)}_{\text{Auxiliary equation}} y = R(x)$$

$$\Rightarrow \quad a_0 m^2 + a_1 m + a_2 = 0$$

→ Roots

1. Real
2. Imaginary
3. Irrational

CASE 1:

If the roots of the auxiliary equation are real and distinct

$$a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + a_{n-2} \frac{d^{n-2}}{dx^{n-2}} y + \dots + a_0 y = R(x)$$

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Suppose ;

$$(D^n - m_n)(D^{n-1} - m_{n-1}) \dots (D - m) = 0$$

$$\therefore \quad D - m = 0$$

$$\frac{dy}{dx} - my = 0$$

$$\left(\frac{d}{dx} - m\right)y = 0$$

$$I.F = e^{\int P dx} = e^{-mx}$$

□ If roots are real and distinct

Complementary Function $= c_1 e^{+m_1 x} + c_2 e^{+m_2 x} + \dots + C_n e^{m_n x}$

→ Example :

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$\therefore \quad m^2 y + 6my + 9y = 0$$

$$(m^2 + 6m + 9)y = 0$$

Auxiliary is given as;

$$m^2 + 6m + 9 = 0$$

$$\therefore \quad m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$m = -3, -3$$

$$\therefore \quad C.F = (C_1 + xC_2)e^{-3x}.$$

Example : $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

The auxiliary equation is given as ;

$$m^2 + 7m + 12 = 0$$

$$\therefore \quad m^2 + 4m + 3m + 12 = 0$$

$$m = -4, -3$$

As, the roots are real and distinct, the complementary function is;

$C.F = C_1 e^{-4x} + C_2 e^{-3x}$. Ans.

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Example : $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0$

$\therefore$ $m^2y + 8my + 15y = 0$

$(m^2 + 8m + 15)y = 0$

Hence, the auxiliary equation is given as;

$m^2 + 8m + 15 = 0$

$m^2 + 5m + 3m + 15 = 0$

$m = -5, -3$

As, the roots are real and distinct. The complementary function is given as;

$C.F = c_1 e^{-5x} + c_2 e^{-3x}$. Ans.

CASE 2 :

If the roots are real and repeated

$\Rightarrow$ If $m_1 = m_2$ and $m_3, m_4, \ldots$ are real and distinct roots.

Hence,

Complementary Function (C.F) $= (c_1 + xc_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$.

$\Rightarrow$ If $m_1 = m_2 = m_3$ and $m_4, m_5, \ldots, m_n$ are real and distinct roots.

Complementary Function (C.F) $= (c_1 + xc_2 + x^2 c_3)e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$.

Example : $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

$\therefore$ $m^2y + 10my + 25y = 0$

$(m^2 + 10m + 25)y = 0$

The auxiliary equation is given as;

$m^2 + 10m + 25 = 0$

$m = -5, -5$

$\therefore$ As, the roots are real but repeated, the complementary function is given as;

$C.F = (c_1 + xc_2)e^{-5x}$. Ans.

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→ Case 3 (i): When roots are imaginary

If roots of the auxiliary equation are imaginary, the complementary function is given as;

$$C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) \quad ; \quad z = \alpha + i\beta$$

→ Case 3 (ii): When roots are imaginary but repeated

$$C.F = e^{\alpha x}((c_1 + xc_2)\cos\beta x + (c_3 + xc_4)\sin\beta x)$$

→ Case 4 (i): When roots are irrational

If roots of the auxiliary equation are irrational ($\alpha \pm \sqrt{\beta}$), the complementary function is given as;

$$C.F = e^{\alpha x}(c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta}).$$

→ Case 4 (ii): When roots are irrational and repeated.

$$C.F = e^{\alpha x}((c_1 + xc_2)\cosh\sqrt{\beta} + (c_3 + xc_4)\sinh\sqrt{\beta})$$

Example: $D^4 - n^4 y = 0$

The auxiliary equation is given as;

$$A.E : m^4 - n^4 = 0$$

$$(m^2 - n^2)(m^2 + n^2) = 0$$

$$(m^2 + n^2)(m - n)(m + n) = 0$$

$$\therefore \quad m = n, -n \text{ and } in$$

Hence, $C.F = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}(c_3 \cos nx + c_4 \sin nx)$. Ans

Example: $\frac{d^4 y}{dx^4} + n^4 y = 0$

The auxiliary equation is given as;

$$A.E = m^4 + n^4 = 0$$

$$m^4 + n^4 + 2m^2n^2 - 2m^2n^2 = 0$$

$$(m^2 + n^2)^2 - 2m^2n^2 = 0$$

$$(m^2 + n^2 + 2mn)(m^2 + n^2 - 2mn) = 0$$

$$\therefore \quad m = \frac{\sqrt{2}n \pm \sqrt{2(1) - 4(n^2)}}{2} \quad , \quad m = \frac{-\sqrt{2}n \pm \sqrt{2 - 4(n^2)}}{2}$$

$$= \frac{\sqrt{2} \pm \sqrt{2(1 - 2n^2)}}{2} \qquad m = -\frac{n}{\sqrt{2}} \pm \frac{\sqrt{1 - 2n^2}}{\sqrt{2}}$$

$$= \frac{}{\sqrt{2}} \pm \frac{\sqrt{1 - 2n^2}}{\sqrt{2}}$$

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Complementary function: $e^{n/\sqrt{2}x}\left(c_1\cosh\sqrt{\frac{1-2n^2}{2}}+c_2\sinh\sqrt{\frac{1-2n^2}{2}}\right)+$
$e^{-n/\sqrt{2}x}\left(c_3\cosh\sqrt{\frac{1-2n^2}{2}}+c_4\sinh\sqrt{\frac{1-2n^2}{2}}\right)$. Ans.

Example: $y''-4y'-5y=0$, $y(0)=1$
$y'(0)=2$.

$\therefore$ The auxiliary equation is given as;

$$m^2-4m-5=0$$

$$\therefore \quad m=\frac{+4\pm\sqrt{16-4(-5)}}{2}$$

$$=\frac{4\pm\sqrt{16+20}}{2}$$

$$=\frac{4\pm 6}{2}$$

$$m=5,-1$$

$$\therefore \quad C.F = y(x) = c_1e^{5x}+c_2e^{-1x}$$

$$\therefore \quad y(0)=1$$

$$1=c_1e^{5(0)}+c_2e^{-1(0)}$$

$$1=c_1+c_2 \quad \text{—— (A)}$$

$$y'(x)=5c_1e^{5x}-c_2e^{-x}$$

$$y'(0)=2$$

$$2=5c_1-c_2 \quad \text{—— (B)}$$

Adding (A) and (B)

$$\therefore \quad 3=6c_1$$

$$\boxed{c_1=1/2}$$

and $\boxed{c_2=1/2}$

Ans.

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Example: $y'' + 4y' + 4y = 0$

$\therefore$ The auxiliary equation is given as;

$m^2 + 4m + 4 = 0$

$m^2 + 2m + 2m + 4 = 0$

$m(m+2) + 2(m+2) = 0$

$m = -2, -2$

$\therefore$ $C.F = (c_1 + xc_2)e^{-2x}$. Ans.

Example: $(D^2 - 2D + 4)^2 y = 0$

$\therefore$ $(D^2 - 2D + 4)(D^2 - 2D + 4)y = 0$

The auxiliary equation is given as;

$m^2 - 2m + 4 = 0$, $m^2 - 2m + 4 = 0$

$m = \dfrac{2 \pm \sqrt{4-16}}{2}$ $m = \dfrac{2 \pm \sqrt{4-16}}{2}$

$m = \dfrac{2 \pm \sqrt{12}i}{2}$ $m = \dfrac{2 \pm \sqrt{12}i}{2}$

$m = 1 \pm i\sqrt{3}$ $m = 1 \pm i\sqrt{3}$.

As, the roots are imaginary but are also repeated.
Hence

$C.F = e^{x}\left((c_1 + xc_2)\cos\sqrt{3}x + (c_3 + xc_4)\sin\sqrt{3}x\right)$ Ans.

Example: Solve D.E

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

$\therefore$ The auxiliary equation is given as;

$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

$\therefore$ $(m^2 - 2m + 2)^2 = 0$

$m = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

As, the roots are imaginary and repeated
Hence,

$C.F = e^{x}\left((c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x\right)$. Ans.

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→ Operator

$$L(y) = a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x)$$

$$L(y) = a_0 D^2 + a_1 D + a_2 y = r(x)$$

$$L(y) = [F(D)] = r(x)$$

$$L(y) = [F(D)]^{-1} \cdot r(x) \longrightarrow \text{Functional derivative.}$$

Functional Integral.

∴ Integral Solution: Complementary Function + Particular Integral.

Case 1: If $r(x) = e^{ax}$

$$P.I = \frac{1}{F(D)} \cdot e^{ax}$$

*** $$P.I = \frac{1}{F(a)} \cdot e^{ax}$$

Example: $(D+1)^3 y = e^{-x}$

∴ The auxiliary equation is given as;

$$(m+1)^3 = 0$$

$$\Rightarrow \quad m = -1, -1, -1$$

∴ Complementary Function $= (c_1 + x c_2 + x^2 c_3) e^{-x}$.

$$P.I = \frac{1}{F(D)} \cdot e^{ax}$$

$$= \frac{1}{(D+1)^3} \cdot e^{-x}$$

But, according to the P.I of exponential function.

$$P.I = \frac{1}{(D+1)^3} \cdot e^{ax} = \frac{1}{F(a)} \cdot e^{ax}$$

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But, as $(D-1)^3 = 0$ for $D=1$

$\therefore$ Differentiating w.r.t D and multiplying w x

$$P.I = \frac{x}{3(D-1)^2} . e^{-x}$$

Again, differentiating w.r.t D and multiplying with x.

$$P.I = \frac{x^2}{6(D-1)} . e^{-x} .$$

Again, differentiating w.r.t D and multiplying with x

$$P.I = \frac{x^3}{6} . e^{-x} .$$

Hence,

General Solution : $C.F + P.I$

$$y(x) = (C_1 + xC_2 + x^2C_3) . e^{-x} + \frac{x^3}{6} . e^{-x} . \text{ Ans.}$$

Example : Solve

$$(D-2)^2 y = 17 . e^{2x}$$

$\therefore$ The auxiliary equation is given as ;

$$(m-2)^2 = 0$$

$\Rightarrow$ $m = 2, 2$ As, the roots are real and repeated.

Hence, $C.F = (C_1 + xC_2) . e^{2x}$

and $P.I = \frac{1}{F(D)} . r(x)$

$$= \frac{1}{(D-2)^2} . 17 . e^{2x} = 17 . \frac{1}{(D-2)^2} . e^{2x}$$

As, $(D-2)^2 = 0$ for $a = 2$

$\therefore$ Differentiating w.r.t D and multiplying by x.

$$P.I = \frac{17 . x . e^{2x}}{2(D-2)}$$

Again, differentiating w.r.t D and multiplying by x.

$$P.I = \frac{17 . x^2 . e^{2x}}{2}$$

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General Solution: C.F + P.I

$$y(x) = (C_1 + xC_2)e^{2x} + \frac{17 \cdot x^2 \cdot e^{2x}}{2}. \quad \underline{\underline{Ans.}}$$

Example: $y'' - 2y' - 3y = 3e^{2x}$.

The auxiliary equation is given as;

$$m^2 - 2m - 3 = 0$$

$\therefore$ $m^2 - 3m + m - 3 = 0$

$$m(m-3) + 1(m-3) = 0$$

$$m = -1, 3$$

As, the roots of equation are real and distinct;

Hence,

$$C.F = C_1 e^{-x} + C_2 e^{3x}$$

and

$$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{(D^2 - 2D - 3)} \cdot 3e^{2x}$$

$$P.I = 3 \cdot \frac{1}{D^2 - 2D - 3} \cdot e^{2x}$$

As, $P.I = \frac{1}{F(D)} \cdot e^{ax}$

$\Rightarrow$ $P.I = \frac{1}{F(a)} \cdot e^{ax}$

Hence

$$P.I = 3 \cdot \frac{e^{2x}}{4 - 4 - 3} = -e^{2x}$$

$\therefore$ The general solution: $C_1 e^{-x} + C_2 e^{3x} - e^{2x}$. $\underline{\underline{Ans.}}$

Example: $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$

The auxiliary equation is given as;

$$m^3 - 2m^2 - 5m + 6 = 0$$

$$(m-1)(m^2 - m - 6) = 0$$

$$(m-1)(m^2 - 3m + 2m - 6) = 0$$

$$(m-1)(m(m-3) + 2(m-3)) = 0$$

$$(m-1)((m+2)(m-3)) = 0$$

$\therefore$ $m = 1, 3, -2$

$1 - 2 - 5 + 6$

$$\begin{array}{r|l} & m^2 - m - 6 \\ \hline m-1 & m^3 - 2m^2 - 5m + 6 \\ & m^3 - m^2 \\ & - \quad + \\ \hline & -m^2 - 5m \\ & -m^2 + m \\ & + \quad - \\ \hline & -6m + 6 \end{array}$$

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As, the roots are real and distinct;

Hence,

$$C.F = c_1 e^{x} + c_2 e^{3x} + c_2 e^{-2x}.$$

and $P.I = \frac{1}{F(D)} \cdot r(x)$

$$= \frac{1}{D^3 - 2D^2 - 5D + 6} \cdot (4e^{-x} - e^{2x})$$

$$= 4 \cdot \frac{e^{-x}}{D^3 - 2D^2 - 5D + 6} - \frac{e^{2x}}{D^3 - 2D^2 - 5D + 6}$$ Replacing $D \to a$

$$= \frac{4 \cdot e^{-x}}{-1 - 2 + 5 + 6} - \frac{e^{2x}}{8 - 8 - 10 + 6}$$

$$= \frac{4 \cdot e^{-x}}{8} - \frac{e^{2x}}{(-4)}$$

$$P.I = \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$ Ans

$\therefore$ The general solution is

$$G.S : C.F + P.I$$

$$: c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$ Ans.

Case 2: If $r(x) = \cos ax$ or $\sin ax$

$$P.I = \frac{1}{F(D)} \cdot r(x)$$ $D^2 \to -a^2$.

$$= [F(D)]^{-1} \cdot \cos ax$$

$$P.I = \frac{1}{F(-a^2)} \cdot \cos ax = \frac{1}{F(-a^2)} \cdot \sin ax$$

→ Solve:

$$y'' + 4y = 6\cos 2x$$

$\therefore$ The auxiliary equation is given as;

$$m^2 + 4 = 0$$

$$\therefore m = \pm 2i$$

As, the roots are imaginary, hence

$$C.F = e^{0x}(c_1 \cos 2x + c_2 \sin 2x).$$ Ans

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$$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{D^2+4} \cdot 6 \cdot \cos 2x$$

As, $D^2 + 4 = 0$, while putting $D^2 = -4$

$\therefore$ Differentiating P.I w.r.t D and multiplying by D.

$$P.I = \frac{1}{2D} \cdot x \cdot 6 \cdot \cos 2x$$

$$= \frac{x}{2} \cdot 6 \cdot \left(\frac{1}{D}\right) \cos 2x$$

$$= 3x \cdot \int \cos 2x \, dx$$

$$= 3x \cdot \frac{\sin 2x}{2}$$

$$P.I = \frac{3}{2} x \sin 2x$$

$\therefore$ The general solution:

$$G.S: c_1 \cos 2x + c_2 \sin 2x + \frac{3}{2} x \sin 2x.$$ Ans.

Example: $4y'' - 4y' + y = \sin 3x$

$\therefore$ The auxiliary equation is given as

$$4m^2 - 4m + 1 = 0$$

$$4m^2 - 2m - 2m + 1 = 0$$

$$2m(2m-1) - 1(2m-1) = 0$$

$$\therefore \quad m = 1/2, 1/2$$

As, the roots are real and repeated, hence

$$C.F = (c_1 + x c_2) \cdot e^{1/2 x}$$

and $$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{4D^2 - 4D + 1} \cdot \sin 3x$$

$$= \frac{1}{-36 - 4D + 1} \cdot \sin 3x$$

$$= \frac{-1}{4D + 35} \cdot \sin 3x$$

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$$P.I = -\frac{1}{4}\cdot\frac{1}{\left(D+\frac{35}{4}\right)}\cdot \sin 3x$$

$$= -\frac{1}{4}\times\frac{\left(D-\frac{35}{4}\right)}{D^2-\left(\frac{35}{4}\right)^2}\cdot \sin 3x$$

$$= -\frac{1}{4}\times\frac{\left(D-\frac{35}{4}\right)}{-9-\left(\frac{35}{4}\right)^2}\cdot \sin 3x$$

$$= \frac{-\frac{1}{\cancel{4}}\times\left(D-\frac{35}{4}\right)}{\frac{(-9\times16)-(35)^2}{\cancel{4}}}$$

$$= \frac{D-\frac{35}{4}}{144+(35)^2}\cdot \sin 3x$$

$$P.I = \frac{3\cos 3x - \frac{35}{4}\sin 3x}{1369}$$

Hence, the general solution;

$$G.S = C.F + P.I$$

$$= (C_1 + C_2 x)e^{x/2} + \frac{3}{1369}\cos 3x - \frac{35}{5476}\cdot \sin 3x. \quad \underline{\underline{Ans}}.$$

Q. Solve

$$(D^2+9)y = 6\sin 3x$$

$\therefore$ The auxiliary equation is given as;

$$m^2+9=0$$

$$\therefore m = 0 \pm 3i = \alpha \pm i\beta$$

Hence, as they roots are imaginary;

$$C.F = e^{\alpha x}(c_1\cos\beta x + c_2 \sin\beta x)$$

$$= e^{0x}(c_1\cos 3x + c_2\sin 3x)$$

$$\Rightarrow C.F = c_1\cos 3x + c_2 \sin 3x.$$

and $P.I = \frac{1}{F(D)}\cdot x(x).$

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$\therefore \quad P.I = \frac{1}{D^2+9} \cdot 6\sin 3x$

On replacing $D^2 \to -(3)^2$, we get $D^2+9=0$

$\therefore$ Differentiating w.r.t D and multiplying by x

$\Rightarrow \quad P.I = \frac{x}{2D} \cdot 6 \cdot \sin 3x$

$$= 3x \cdot \left(\frac{1}{D}\right) \cdot \sin 3x$$

$$= 3x \int \sin 3x \cdot dx$$

$$P.I = 3x \times \frac{(-\cos 3x)}{3} = -x\cos 3x$$

$\therefore$ The general solution is given as;

$$G.S = P.I + C.F$$

$$= c_1\cos 3x + c_2 \sin 3x - x\cos 3x$$

$$G.S = (c_1 - x)\cos 3x + c_2 \sin 3x \quad \text{Ans.}$$

Q. Solve

$$y'' - 3y' - 3y = (-2)\cos 3x$$

$\therefore$ The auxiliary equation is given as;

$$m^2 - 3m - 3 = 0$$

$$\therefore \quad m = \frac{3 \pm \sqrt{9+12}}{2}$$

$$m = \frac{3 \pm \sqrt{21}}{2} = \alpha \pm \sqrt{\beta}$$

As, the roots are irrational

$$C.F = e^{\alpha x}(c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta})$$

$$= e^{3/2 x}\left(c_1 \cosh\sqrt{\frac{21}{4}} + c_2 \sinh\sqrt{\frac{21}{4}}\right)$$

and $PI = \frac{1}{F(D)} \cdot r(x)$

$$= \frac{1}{D^2 - 3D - 3} \cdot (-2)\cos 3x$$

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$\therefore$ Putting $D^2 \to -9$

$$\begin{array}{r} 144 \\ 81 \\ \hline 225 \end{array}$$

$\therefore$ $P.I = (+2) \times \frac{1}{+12+3D} \cdot \cos 3x$

$= (2) \times \frac{(3D-12)}{9D^2-144} \cdot \cos 3x$

$= (2) \times \frac{(3D-12)}{-81-144} \cdot \cos 3x$

$= \left(\frac{2}{-225}\right)\left(3D\cos 3x - 12\cos 3x\right)$

$= \left(\frac{-2}{225}\right)\left(9(-\sin 3x) - 12\cos 3x\right)$

$\Rightarrow P.I = \frac{18}{225}\sin 3x + \frac{24}{225}\cos 3x$

$\therefore$ The general solution is given as;

$G.S = P.I + C.F$

$y = G.S = e^{3/2 x}\left(c_1 \cosh\sqrt{\frac{21}{4}} + c_2 \sinh\sqrt{\frac{21}{4}}\right) + \frac{18}{225}\sin 3x + \frac{24}{225}\cos 3x$

Ans.

Q. $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$

$\therefore$ The auxiliary equation will be given as;

$m^2 - 4m + 1 = 0$

$\therefore$ $m = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm \sqrt{12}}{2}$

$m = 2 \pm \sqrt{3} = \alpha \pm \sqrt{\beta}$

As, the roots are irrational,

$C.F = e^{\alpha x}(c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta})$

$C.F = e^{2x}(c_1 \cosh\sqrt{3} + c_2 \sinh\sqrt{3})$

$\therefore$ $P.I = \frac{1}{F(D)} \cdot R(x)$

$= \left(\frac{1}{D^2-4D+1}\right)\left(\cos x \cos 2x + \sin^2 x\right)$

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$$P.I = \left(\frac{1}{D^2-4D+1}\right)(\cos 2x \cos x) + \left(\frac{1}{D^2-4D+1}\right)(\sin^2 x)$$

$$= \left(\frac{1}{D^2-4D+1}\right)(\cos 3x + \cos x) + \left(\frac{1}{D^2-4D+1}\right)\left(\frac{1}{2} - \frac{\cos 2x}{2}\right)$$

$$= \left(\frac{\cos 3x}{-8-4D}\right) + \left(\frac{\cos x}{-4D}\right) + \left(\frac{e^0}{1}\right)\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{(\cos 2x)}{-3-4D}\right)$$

$$= -\frac{(D-2)\cos 3x}{4(D^2-4)} + \left(-\frac{1}{4}\right)\sin x + \frac{1}{2} + \frac{1}{8}\left(\frac{D-3/4}{D^2-3/4}\right)\cos 2x$$

$$= \left(\frac{D\cos 3x - 2\cos 3x}{4 \times +13}\right) - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{D\cos 2x - 3/4\cos 2x}{-39/4}\right)$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{+2\sin 2x + \frac{3}{4}\cos 2x}{+39/4}\right)$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{8\sin 2x + 3\cos 2x}{39}\right)$$

$\therefore$ General solution is given as;

$$GS = PI + CF$$

$$= \frac{-3\sin 3x - 2\cos 3x}{52} - \frac{\sin x}{4} + \frac{1}{2} + \frac{1}{8}\left(\frac{8\sin 2x}{39} + \frac{3\cos 2x}{39}\right) +$$

$$e^{2x}(c_1 \cosh\sqrt{3} + c_2 \cosh\sqrt{3}) \quad \text{Ans.}$$

Case 3: If $h(x) = x^n$

$$P.I = \frac{1}{F(D)} \cdot x^n$$

$$P.I = [F(D)]^{-1} \cdot x^n$$

$\hookrightarrow$ Binomial expansion

Q. Solve

$$y'' + 16y = 64x^2$$

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The auxiliary equation is given as;

$$m^2 + 16 = 0$$

$$\therefore \quad m = \pm 4i = \alpha \pm \beta i$$

As, the roots are imaginary.

Hence;

$$C.F = c_1 \cos 4x + c_2 \sin 4x$$

$$P.I = \frac{1}{F(D)} \cdot r(x)$$

$$= \frac{1}{D^2+16} \cdot 64x^2$$

$$= \frac{1}{16} \cdot 64 \cdot \left[1 + \left(\frac{D}{4}\right)^2\right]^{-1} \cdot x^2$$

$$= 4 \cdot \left[1 + \frac{D}{4} + \frac{D^2}{16} - \cdots\right] x^2$$

$$= 4\left[x^2 + \frac{2x}{4} + \frac{2}{16}\right]$$

$$= 4x^2 + 2x + \frac{1}{2} \quad \text{Ans.}$$

Q. Solve

$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$$

$$\therefore \quad m^4 + 3m^2 = 0$$

$$m^2(m^2+3) = 0$$

$$\therefore \quad m = 0, 0, \pm\sqrt{3}i$$

Hence, the complementary function is given as;

$$C.F = (c_1 + x c_2) + (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$$

and $P.I = \frac{1}{F(D)} \cdot r(x)$

$$= \frac{1}{D^4+3D^2} \cdot 108x^2$$

$$= \frac{1}{D^2(D^2 \quad)} \cdot 108x^2 = \frac{1}{3}\left[\frac{D^2+3-D^2}{D^2(D^2+3)}\right] \cdot 108x^2$$

$$\cancel{= \frac{1}{(1+(\frac{D}{\sqrt{3}})^2)} \cdot 108x^2} = \frac{1}{3}\left[\frac{1}{D^2} - \frac{1}{D^2+3}\right] \cdot 108x^2$$

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$$= \frac{1}{3}\left[\frac{1}{D^2}\cdot 108x^2 - \frac{1}{D^2+3}\cdot 108x^2\right]$$

$$= \frac{1}{3}\left[108\iint x^2\,dx\,dx - \frac{1}{3\left(1+\left(\frac{D}{\sqrt{3}}\right)^2\right)}\cdot 108x^2\right]$$

$$= \frac{1}{3}\left[\frac{108}{12}x^4 - \frac{1}{3}\left[1+\left(\frac{D}{\sqrt{3}}\right)^2\right]\cdot 108x^2\right]$$

$$= 3x^4 - \frac{1}{9}\left[1-\frac{D^2}{3}\right]108x^2$$

$$= 3x^4 - \frac{1}{9}\left[108x^2 - \frac{2\times 108}{3}\right]$$

$$= 3x^4 - 12x^2 + \frac{2\times 108}{3\times 9}$$

$$P.I = 3x^4 - 12x^2 + 8$$

Hence, the general solution is given as;

$$GS = P.I + CF$$

$$= (C_1 + xC_2) + (c_3\cos\sqrt{3}x + c_4\sin\sqrt{3}x) + 3x^4 - 12x^2 + 8$$

Ans.

Q. Solve D.E

$$(D^3 - 7D^2 + 6)y = 1 + x^2$$

∴ The auxiliary equation can be given as;

$$D^3 - 7D^2 + 6 = 0$$

$$\therefore \quad m^3 - 7m^2 + 6 = 0$$

$$\Rightarrow (m-1)(m^2-6m-6) = 0$$

$$\therefore \quad m = 1, \frac{6 \pm \sqrt{36+24}}{2}$$

$$m = 1, 3 \pm \sqrt{15}$$

$$\begin{array}{r|l} & m^2 - 6m - 6 \\ \hline (m-1) & m^3 - 7m^2 + 6 \\ & m^3 - m^2 \\ & - \quad + \\ \hline & -6m^2 + 6 \\ & -6m^2 + 6m \\ & + \quad + \\ \hline & -6m + 6 \\ & -6m + 6 \\ & + \quad - \end{array}$$

Hence, the complementary function CF can be given as;

$$CF = c_1e^x + e^{3x}(c_2\cosh\sqrt{15} + c_3\sinh\sqrt{15}).$$

and $P.I = \frac{1}{F(D)}\cdot r(x)$

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$$P.I = \frac{1}{(D^3-7D^2+6)} \cdot (1+x^2)$$

$$= \left(\frac{1}{D^3-7D^2+6} \cdot e^{0x}\right) - \left(\frac{1}{D^3-7D^2+6}\right) x^2$$

$$= \frac{1}{6} - \frac{1}{6}\left[1+ \frac{D^3-7D^2}{6}\right]^{-1} \cdot x^2$$

$$(1+ax)^{-1} = 1 - ax + \frac{a(a+1)}{2!}x^2 - \frac{a(a+1)(a+2)}{3!}x^3 + \dots$$

$$\therefore PI = \frac{1}{6} - \frac{1}{6}\left[1 - \frac{D^3-7D^2}{6}\right] \cdot x^2$$

$$= \frac{1}{6} - \frac{1}{6}\left[x^2 - 0 + \frac{7 \times 2}{6}\right]$$

$$P.I = \frac{1-x^2-14}{6}$$

$\therefore$ The general solution is given by;

$$GS = CF + PI$$

$$GS = c_1 e^x + e^{3x}(c_2 \cosh\sqrt{15} + c_3 \cosh\sqrt{15}) + \frac{1-x^2-14}{6}$$

Ans.

Q. Solve:

$$(D^3 - D^2 - 6D)y = 1 + x^3$$

$\therefore$ The auxiliary equation is given as;

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m^2 - 3m + 2m - 6) = 0$$

$$m(m(m-3) + 2(m-3)) = 0$$

$$m(m+2)(m-3) = 0$$

$$\Rightarrow m = 0, -2, 3$$

As, ~~the~~ roots are real and distinct;

$$\Rightarrow CF = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$\therefore PI = \frac{1}{F(D)} \cdot r(x) = \frac{1}{(D^3-D^2-6D)} \cdot (1+x^3)$$

$$= \left[\frac{1}{D^3-D^2-6D}\right] + \frac{1}{(-6D)} \cdot \left[1 - \left(\frac{D^2-D}{6}\right)\right]^{-1} \cdot x^3$$

$$= \left[\frac{x}{3D^2-2D-6} \cdot e^{0x}\right] - \frac{1}{6D}\left[1 + \frac{D^2-D}{6} + \frac{D^4+D^2-2D^3}{36}\right] \cdot x^3$$

$$= \left(\frac{x}{-6}\right) - \frac{1}{6D}\left[x^3 + \frac{6x}{6} - \frac{3x^2}{6} + 0 + \frac{6x}{36} - \frac{12}{36}\right]$$

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$$P.I = -\frac{x}{6} - \frac{x^4}{24} - \frac{x^2}{12} + \frac{x^3}{36} - \frac{x^2}{72} + \frac{1}{18}x$$

$\frac{x-3x}{18}$

$$P.I = -\frac{x}{9} - \frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72}$$

Hence, the general solution is given as;

$$GS = PI + CF$$

$$= c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72} - \frac{x}{9}. \quad \text{Ans.}$$

→ Case 4: If V is any function

$$P.I = \frac{1}{F(D)} \cdot e^{ax} \cdot V$$

$$P.I = e^{ax} \cdot \frac{1}{F(D+a)} \cdot V$$

Q. Solve D.E

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = (1-x)e^{2x}$$

∴ The auxiliary equation is given as;

$$m^2 - 7m + 12 = 0$$

$$\Rightarrow m^2 - \cancel{7}3m - 4m + 12 = 0$$

$$m(m-3) - 4(m-3) = 0$$

$$(m-3)(m-4) = 0$$

$$\Rightarrow m = 3, 4.$$

∴ As, the roots are real and distinct;

$$C.F = c_1 e^{3x} + c_2 e^{4x}.$$

$$\therefore PI = \left[\frac{1}{D^2 - 7D + 12}\right] \cdot e^{2x}(1-x)$$

$$= e^{2x} \cdot \frac{1}{D^2 + 4 + 4D - 7D - 14 + 12} \cdot (1-x)$$

$$= e^{2x} \cdot \frac{1}{D^2 - 3D + 2} \cdot (1-x)$$

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$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[1+\frac{D^2-3D}{2}\right]^{-1}\cdot x\right.$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[1-\frac{D^2+3D}{2}\right]\cdot x\right.$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}\left[x - \frac{0}{2} + \frac{3}{2}\right]\right]$$

$$= e^{2x}\left[\frac{1}{2} - \frac{1}{2}x - \frac{3}{4}\right]. \text{ Ans.}$$

Hence, General solution is given as;

$$GS = PI + CF \qquad \text{Ans.}$$

$$= c_1 e^{3x} + c_2 e^{4x} + e^{2x}\left[\frac{1}{2} - \frac{x}{2} - \frac{3}{4}\right].$$

Q. Solve D.E

$$y'' - 2y' + 2y = e^{2x}\cos 2x$$

∴ The auxiliary equation can be given as;

$$m^2 - 2m + 2 = 0$$

$$m^2 = \frac{+2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$m = 1 \pm 2i = \alpha \pm i\beta$$

As, the roots are imaginary;

$$C.F = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$= e^{x}(c_1\cos 2x + c_2\sin 2x).$$

$$\therefore \quad P.I = \frac{1}{F(D)}\cdot R(x)$$

$$= \frac{1}{D^2-2D+2}\cdot e^{2x}\cos 2x$$

$$= e^{2x}\cdot\frac{1}{(D+2)^2-2(D+2)+2}\cdot\cos 2x$$

$$= e^{2x}\cdot\frac{1}{D^2+4+4D-2D-4+2}\cdot\cos 2x$$

$$= e^{2x}\cdot\frac{1}{-4+\cancel{4}+2D-\cancel{4}+2}\cdot\cos 2x = e^{2x}\cdot\frac{1}{2D-2}\cdot\cos 2x$$

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$$= \frac{e^{2x}}{2} \cdot \frac{D+1}{D^2-1} \cdot \cos 2x$$

$$= \frac{e^{2x}}{2} \cdot \frac{(D\cos 2x + \cos 2x)}{-5}$$

$$= \frac{e^{2x}}{2} \cdot \left(\frac{-2\sin 2x + \cos 2x}{-5} \right). \quad \text{Ans.}$$

$\therefore$ The general solution is given as;

$$GS = CF + PI$$

$$= e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{e^{2x}}{10}(2\sin 2x - \cos 2x)$$

Ans.

Q. Solve

$$D^2 - 4D + 5y = e^{x} \cos \frac{x}{2}.$$

$\therefore$ The auxiliary equation is given as;

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16-20}}{2}$$

$$m = \frac{4 \pm 2i}{2} = 2 \pm i = \alpha \pm i\beta$$

$$\therefore \quad C.F = e^{2x}(c_1 \cos x + c_2 \sin x)$$

$$P.I = \frac{1}{F(D)} \cdot h(x)$$

$$= \frac{1}{D^2 - 4D + 5} \cdot e^{x} \cdot \cos \frac{x}{2}$$

$$= e^{x} \cdot \frac{1}{D^2 + 1 + 2D - 4D - 4 + 5} \cdot \cos \frac{x}{2}$$

$$= e^{x} \cdot \frac{1}{-\frac{1}{4} + 2 - 2D} \cdot \cos \frac{x}{2}$$

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$$4e^{x}\cdot\frac{1}{7-8D}\cdot\cos\frac{x}{2}$$ $\frac{49}{\frac{16}{5}}$

$$\frac{4}{(-8)}\cdot e^{x}\cdot\frac{1}{D-\frac{7}{8}}\cdot\cos\frac{x}{2}$$

$$-2\cdot e^{x}\cdot\frac{D+\frac{7}{8}}{-\frac{1}{4}-\frac{49}{64}}\cdot\cos\frac{x}{2}$$

$$\frac{-2e^{x}}{\left(\frac{-16-49}{64}\right)}\cdot\left(\frac{-\sin\frac{x}{2}}{2}+\frac{7}{8}\cos\frac{x}{2}\right)$$

$$=\frac{128}{65}e^{x}\cdot\left[\frac{7}{8}\cos\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}\right]\cdot \text{Ans}$$

Hence, the general solution is given as;

$$GS = CF + PI$$

$$= e^{2x}(c_1\cos x + c_2\sin x) + \frac{128}{65}e^{x}\cdot\left[\frac{7}{8}\cos\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}\right]\cdot \text{Ans.}$$

Euler-Cauchy differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = r(x)$$

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = r(x)$$

$$\underbrace{a_0, a_1, a_2 \text{ are constants}}$$

Linear differential equation with constant coefficents.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = r(x)$$

Linear non-homogenous differential equations.

=> Method to solve Euler-Cauchy differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = r(x)$$

$$x\frac{dy}{dx} \longrightarrow \text{constant coefficient}$$

$$x = e^{z}, \quad z = \ln x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{x} = \frac{1}{x}\frac{dy}{dz}. \quad \Rightarrow \frac{dy}{dz} = \frac{x\,dy}{dx}.$$

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$\therefore \quad x\frac{dy}{dx} \equiv Dy, \quad D \equiv \frac{d}{dz}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$= \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right]$

By using product rule,

$= \frac{dy}{dz}\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dz}{dx}$

$= \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d^2y}{dx^2}\cdot\frac{dz}{dx}$

$= \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = D^2y - Dy.$

Q. $(D(D-1)+2D+2)y = 10\left(e^z + \frac{1}{e^z}\right)$

$\therefore \quad (D^2+D+2)y = 10\left(e^z + \frac{1}{e^z}\right)$

The auxiliary equation is given as;

$m^2 + m + 2 = 0$

$\therefore \quad m = \frac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

Hence,

$C.F = e^{-1/2z}\left(c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right)$

$\therefore \quad P.I = \frac{1}{F(D)}\cdot r(x)$

$= \frac{1}{D^2+D+2}(e^z+e^{-z}) = \frac{1}{4}e^z + \frac{1}{2}e^{-z}$

$\Rightarrow G.S = e^{-1/2z}\left(c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right) + \frac{1}{4}e^z + \frac{1}{2}e^{-z}$ Ans.

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Q. $D(D-1)(D-2)y + 3D(D-1)y + Dy + y = e^z + z$

$(D^3 - \cancel{2D^2} - \cancel{D^2} + \cancel{2D} + 3\cancel{D^2} - \cancel{3D} + \cancel{D} + 1)y = e^z + z$

$\Rightarrow \quad (D^3+1)y = e^z + z$

Hence, the auxiliary equation is given as:

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

Hence,

$$CF = c_1 e^{-z} + e^{-1/2 z}\left[c_1 \cos\frac{\sqrt{3}}{2}z + c_2 \sin\frac{\sqrt{3}}{2}z\right]$$

$$\Rightarrow \quad P.I = \frac{1}{F(D)} . e^z . z$$

$$= \frac{1}{D^3+1} . (e^z + z)$$

$$= \frac{1}{2}e^z + [1 - D^3]z$$

$$P.I = \frac{e^z}{2} + z$$

Hence:

$$GS = CF + PI$$

$$GS = c_1 e^{-z} + e^{-z/2}\left[c_1 \cos\frac{\sqrt{3}}{2}z + c_2 \sin\frac{\sqrt{3}}{2}z\right] + \frac{e^z}{2} + z \text{ . Ans.}$$

→ Method of variation of parameters

Consider the second order differential equation:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x)$$

(a) Complementary function: $\quad R = \frac{r(x)}{a_0}$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore \quad y = A(x) e^{m_1 x} + B(x) e^{m_2 x}$$

$$y = A(x).u + B(x)v$$

$$\therefore \quad A(x) = -\int \frac{Rv}{W} dx + a$$

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$$B(x) = +\int \frac{Ru}{w}\,dx + b$$

$W \rightarrow$ Wronskian

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

Q. Solve the D.E

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

$\therefore$ The auxiliary equation is given as:

$$m^2 + a^2 = 0$$
$$m = \pm ai$$

$\therefore$ $C.F = c_1 \cos ax + c_2 \sin ax$

$\Rightarrow$ $y = A(x)\cos ax + B(x)\sin ax$ $\qquad R = \sec ax$

$y = A(x)u + B(x)v$

$\Rightarrow$ $u = \cos ax$, $v = \sin ax$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$\therefore$ $A(x) = -\int \frac{\sec ax . \sin ax}{a} . dx + a'$

$$= \frac{-1}{a}\int \tan ax\, dx + a'$$

$$A(x) = -\frac{1}{a}\ln|\sec x| + a'$$

and $B(x) = \int \frac{Ru}{w} dx + b' = \int \frac{\sec ax . \cos ax}{a} dx + b'$

$$B(x) = \frac{x}{a} + b'$$

Hence:

$$y = +\left(\frac{1}{a}\ln|\cos x| + a'\right)\cos ax + \left(\frac{x}{a} + b'\right)\sin ax.$$ Ans.

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Q. Solve the following differential equations by variation of parameters.

(a) $\frac{d^2y}{dx^2} + y = \tan x$

$\because$ The auxiliary equation is given as;

$m^2 + 1 = 0$

$m = \pm i$

$\therefore C.F = C_1 \cos x + C_2 \sin x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\Rightarrow y = A(x) \cos x + B(x) \sin x$

$y = A(x) u + B(x) v$

$\Rightarrow u = \cos x$ and $v = \sin x$

$\therefore A(x) = -\int \frac{Rv}{W} dx + a$

$= -\int \frac{\tan x \cdot \sin x}{1} \cdot dx + a$

$= -\int \frac{\sin^2 x}{\cos x} \cdot dx$

$= -\int \frac{1 - \cos^2 x}{\cos x} dx$

$= -\int \sec x \, dx + \int \cos x \, dx$

$A(x) = -\ln|\sec x + \tan x| + \sin x + a$

and $B(x) = \int \frac{Ru}{W} dx + b$

$= \int \frac{\tan x \cdot \cos x}{1} dx + b$

$= \int \sin x \, dx + b$

$= -\cos x + b$

$\Rightarrow y = (-\ln|\sec x + \tan x| + \sin x + a) \cos x + (-\cos x + b) \sin x$

Ans.

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(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$

$\therefore$ The auxiliary equation is given as;

$m^2 - 2m = 0$ $\qquad$ $R = e^x \sin x$

$m(m-2) = 0$

$\Rightarrow$ $C.F = c_1 + c_2 e^{2x}$

Hence, $y = A(x).1 + B(x).e^{2x}$

$y = A(x)u + B(x)v$ $\qquad$ $W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$

$u = 1, \quad v = e^{2x}$

$\therefore$ $A(x) = -\int \frac{Rv}{W} dx + a$

$= -\int \frac{e^x \sin x . \cancel{e^{2x}}}{2\cancel{e^{2x}}} + a$

$A(x) = -\frac{1}{2}\int e^x \sin x dx + a$

Let,

$I = \int \underset{II}{e^x} \underset{I}{\sin x} dx = \sin x e^x - \int \underset{I}{\cos x} \underset{II}{e^x} dx$

$I = \sin x e^x - \cos x e^x + \int (-\sin x) e^x dx$

$\Rightarrow$ $I = \frac{e^x(\sin x - \cos x)}{2}$

$\Rightarrow$ $A(x) = \frac{e^x(\cos x - \sin x)}{4} + a$

and $B(x) = \int \frac{Ru}{W} dx = \int \frac{e^x \sin x}{e^{2x}} . dx$

$B(x) = \int e^{-x} \sin x dx$

$\therefore$ $I = \int \underset{II}{e^{-x}} \underset{I}{\sin x} dx$

$= -\sin x e^{-x} - \int (+\underset{I}{\cos x}).(\underset{II}{+e^{-x}}).dx$

$= -\sin x e^{-x} - \left[\cos x(-e^{-x}) - \int \sin x(-e^{-x}) dx\right]$

$\therefore$ $I = \frac{-e^{-x}(\sin x + \cos x)}{2} + b = B(x).$

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$\therefore$ Hence,

$$y = A(x) + B(x)e^{2x}$$
$$= \left(\frac{e^{x}}{4}(\cos x - \sin x) + a\right) + \left(b - \frac{e^{-x}}{2}(\sin x + \cos x)\right)e^{2x}.$$

Ans.

Q. By using the method of variation of parameter solve the following differential equation.

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x$$

$\therefore$ Using Euler-Cauchy differential equation;

$$x = e^z$$
$$z = \ln x.$$

$\therefore$ $D(D-1)y + 2Dy - 12y = x^3 \log x$

$\Rightarrow$ $D^2y + Dy - 12y = x^3 \log x$

$\therefore$ The auxiliary equation is given as;

$$m^2 + m - 12 = 0$$
$$m^2 + 4m - 3m - 12 = 0$$
$$m(m+4) - 3(m+4) = 0$$
$$(m-3)(m+4) = 0$$

$$W = \begin{vmatrix} e^{3z} & e^{-4z} \\ 3e^{3z} & -4e^{-4z} \end{vmatrix}$$
$$= -4e^{-z} - 3e^{-z}$$
$$W = -7e^{-z}$$

$$C.F = c_1 e^{3z} + c_2 e^{-4z}$$

$$y = A(z)e^{3z} + B(z)e^{-4z}$$
$$y = A(z)u + B(z)v$$

$R = x\log x$
$= e^z . z$

$\Rightarrow$ $u = e^{3z}$, $v = e^{-4z}$

$x = e^z$
$dx = e^z dz$

$\Rightarrow$ $A(x) = -\int \frac{Rv}{W} dx + a$

$$A(z) = +\int \frac{e^z z . e^{-4z}}{+7e^{-z}} dx + a$$

$$A(z) = \frac{1}{7}\int e^{2z-4z+z} z . dz + a$$

$$A(z) = \frac{1}{7}\int \underset{II}{e^{-z}} . \underset{I}{z} . dz + a$$

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$$A(z) = \frac{1}{7}\left[-z e^{-z} + \int e^{-z} dz\right] + a$$

$$A(z) = \frac{1}{7}\left[-z e^{-z} - e^{-z}\right] = -\frac{e^{-z}}{7}[z+1] + a$$

and

$$B(z) = \int \frac{Ru}{w} dx + b$$

$$= \int \frac{e^{z} \cdot z \cdot e^{3z}}{-7e^{-z}} \cdot e^{z} dz + b$$

$$= \int \frac{e^{z+3z+z+z}}{-7} \cdot z \, dz + b$$

$$= \frac{-1}{7} \int \underset{II}{e^{6z}} \cdot \underset{I}{z} \, dz + b$$

$$\therefore B(z) = -\frac{1}{7}\left[\frac{z e^{6z}}{6} - \int \frac{e^{6z}}{6} dz\right] + b$$

$$= -\frac{1}{7}\left[\frac{z e^{6z}}{6} - \frac{e^{6z}}{36}\right] + b$$

$$\therefore y(z) = \left(-\frac{e^{-z}}{7}[z+1] + a\right)e^{3z} + \left(\frac{e^{6z}}{42}\left[\frac{1}{6} - z\right] + b\right)e^{-4z}$$

$$y(x) = \left(-\frac{e^{-\ln x}}{7}[\ln x + 1] + a\right)e^{3\ln x} + \left(\frac{e^{6\ln x}}{42}\left[\frac{1}{6} - \ln x\right] + b\right)e^{-4\ln x}$$

$$y(x) = \left(\frac{-1}{7x}[\ln x + 1] + a\right)x^{3} + \left(\frac{x^{6}}{42}\left[\frac{1}{6} - \ln x\right] + b\right)\frac{1}{x^{4}} \quad \text{Ans.}$$

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→ To find the complete solution of the differential equation.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

by the method of change of independent variable.

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{dy}{dz}\right)\left(\frac{d^2z}{dx^2}\right)$$

∴ Substituting $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in given differential equation;

We get

$$\frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \left[\frac{Q}{\left(\frac{dz}{dx}\right)^2}\right]y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \quad \text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

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Differential equation obtained after changing the independent variable from 'x' to 'z':

$$\therefore \quad \frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

Q. Solve the D.E :

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 2x^2 y = x^4$$

Hence,

$$P = -1/x$$

$$Q = 2x^2$$

and $R = x^4$

$$\Rightarrow \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

Let $\boxed{Q_1 = 1}$;

$$\left(\frac{dz}{dx}\right)^2 = 2x^2$$

$$\boxed{\frac{dz}{dx} = \sqrt{2}x}$$

$$\therefore \quad \int dz = \int \sqrt{2}x \, dx$$

$$\boxed{z = \frac{x^2}{\sqrt{2}}},$$

and

$$\boxed{\frac{d^2z}{dx^2} = \sqrt{2}}$$

Hence, we get

$$P_1 = \left[\frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)}\right] = \frac{\sqrt{2} - \frac{1}{x}\cdot\sqrt{2}x}{(\sqrt{2}x)^2} = 0$$

$$\therefore \quad R_1 = \frac{x^4}{(\sqrt{2}x)^2} = \frac{x^2}{2}$$

$$D \equiv \frac{d}{dz}$$

Hence, we get

$$D^2y + 0Dy + y = \frac{x^2}{2}.$$

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$\therefore$ The auxiliary equation is given as;

$$m^2 + 1 = 0$$

$$\Rightarrow \quad m = \pm i$$

Hence $CF = C_1 \cos z + C_2 \sin z = C_1 \cos\left(\frac{\sqrt{2}x^2}{2}\right) + C_2 \sin \frac{x^2}{\sqrt{2}}$

and $PI = \frac{1}{F(D)} \cdot R(z)$

$$= \frac{1}{D^2+1} \cdot \frac{z}{\sqrt{2}}$$

$$= [1 + D^2]^{-1} \cdot \frac{z}{\sqrt{2}}$$

$$= [1 - D^2] \frac{z}{\sqrt{2}}$$

$$P.I = \frac{z}{\sqrt{2}} = \frac{x^2}{2}.$$

$\therefore$ The General solution is given as;

$$GS = C.F + P.I$$

$$G.S = C_1 \cos\left(\frac{x^2}{\sqrt{2}}\right) + C_2 \cos\left(\frac{x^2}{\sqrt{2}}\right) + \frac{x^2}{2} \quad \text{Ans.}$$

Q. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = \frac{\cos(\log(1+x))}{(1+x)^2}$

$$\therefore \quad \frac{d^2y}{dx^2} + \left(\frac{1}{1+x}\right) \frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{\cos(\log(1+x))}{(1+x)^4}$$

$$\therefore \quad P = \frac{1}{1+x}$$

$$Q = \frac{1}{(1+x)^2}$$

$$R = \frac{\cos(\log(1+x))}{(1+x)^4}$$

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$\therefore \quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$\therefore \quad \left(\dfrac{dz}{dx}\right)^2 \times Q_1 = \dfrac{1}{(1+x)^2}$

Let $\boxed{Q_1 = 1}$

$\dfrac{dz}{dx} = \dfrac{1}{1+x}$ and $\dfrac{d^2z}{dx^2} = \dfrac{-1}{(1+x)^2}$

$\therefore \quad \boxed{z = \ln(1+x)}$

$\therefore \quad P_1 = \dfrac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = \dfrac{\frac{-1}{(1+x)^2} + \left(\frac{-1}{1+x}\right)\left(\frac{1}{1+x}\right)}{\left(\frac{1}{1+x}\right)^2} = 0$

$\Rightarrow \quad \boxed{P_1 = 0}$

and $\quad R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2} = \dfrac{\cos(\log(1+x))}{(1+x)^4 \cdot \left(\frac{1}{1+x}\right)^2} = \dfrac{\cos(\log(1+x))}{(1+x)^2}$

$\therefore \quad D \equiv \dfrac{d}{dz}$

D.E obtained:

$D^2y + 0\,Dy + 1.y = \dfrac{\cos z}{e^{2z}}$

$\therefore$ The auxiliary equation is given as:

$m^2 + 1 = 0$

$m = \pm i$

Hence, $CF = c_1 \cos z + c_2 \sin z$

$CF = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+z))$

Now;

$PI = \dfrac{1}{F(D)} \cdot r(z)$

$= \dfrac{1}{(D^2+1)} \cdot e^{-2z}\cos z$

$= e^{-2z} \cdot \dfrac{1}{(D-2)^2+1} \cdot \cos z$

$= e^{-2z} \cdot \dfrac{1}{D^2+4-4D+1} \cdot \cos z$

$= e^{-2z} \cdot \dfrac{1}{4-4D} \cdot \cos z$

$= \dfrac{e^{-2z}}{-4} \times \dfrac{(D-1)}{(-2)} \cdot \cos z$

$= \dfrac{e^{-2z}}{}(-\sin z - \cos z)$

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$\therefore PI = -\dfrac{(1+x)^{-2}}{8}\left(\sin(\ln(1+x)) + \cos(\ln(1+x))\right)$

$\therefore$ General solution is given as;

$GS = CF + PI$

$y = GS = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) - \dfrac{(1+x)^{-2}}{8}$
$(\sin(\ln(1+x)) + \cos(\ln(1+x)))$

Ans.

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Laplace Transformation

Transformation

Plane to plane (co-ordinate) (x, y, z)

space to space (parameter) (t)

Notation: $L\{f(t)\}$

Laplace transformation of function of f.

→ Formula:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \text{ where } t > 0.$$

$f(t)$ is known as kernal,

$f(t)$: e^{at}, $\sin at$, $\cos at$, t^n.

Example: Find $L\{1\}$.

$\therefore \quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right]$$

$$= -\frac{1}{s}[0 - 1]$$

$$L\{1\} = \frac{1}{s}. \quad \text{Ans.}$$

=> $L\{t^n\} = ?$

Using Laplace transformation:

Let $st = z$

$s\,dt = dz$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \cdot dt = \int_0^\infty e^{-z} \cdot \left(\frac{z}{s}\right)^n \cdot s \cdot dt = \frac{\Gamma(n+1)}{s^{n+1}}.$$

$$= \frac{n!}{s^{n+1}}.$$

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$$L\{t^2\} = \int_0^\infty \underset{II}{e^{-st}} \cdot \underset{I}{t^2} \cdot dt$$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 2\underset{I}{t} \cdot \overset{II}{\frac{e^{-st}}{-s}} dt$$

$$= \left[\frac{t^2}{-s} \left(\left[\lim_{t \to \infty} e^{-st} \right] - 1 \right) \right] + \frac{2}{s} \left[\left[t \cdot \frac{e^{-st}}{s} \right]_0^\infty - \left[\frac{1}{s^2} \cdot e^{-st} \right]_0^\infty \right]$$

$$= \cancel{\frac{t^2}{s}}^{0} + \frac{2}{s} \left[\left(\cancel{\lim_{t \to \infty} \frac{t \cdot e^{-st}}{s}}^{0} - \cancel{\frac{t}{s}}^{0} \right) - \frac{1}{s^2} [0 - 1] \right]$$

$$= \cancel{\frac{t^2}{s}}^{0} + \frac{2}{s} \left[\left(-\cancel{\frac{t}{s}}^{0} \right) + \frac{1}{s^2} \right]$$

$$L\{t^2\} = \frac{2}{s^3} \quad \text{Ans}$$

$\therefore$ If $n \to$ fractional;

$$L\{t^n\} = \frac{\sqrt{(n+1)}}{s^{n+1}} \qquad \text{where } n > 0$$

If $n \to$ integer

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

Q. Find Laplace Transformation of e^{at}.

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[-\frac{e^{-t(s-a)}}{s-a} \right]_0^\infty = \lim_{t \to \infty} \left(-\frac{e^{-\infty(s-a)}}{s-a} \right) + \left(\frac{1}{s-a} \right)$$

$$L\{e^{at}\} = \frac{1}{s-a}.$$

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→ Laplace transformation of $\sin at$.

$\therefore L\{\sin at\} = L\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\}$ $\quad \sin at = \frac{e^{iat}-e^{-iat}}{2i}$

$$= \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{\cancel{s}+ia-\cancel{s}+ia}{s^2-(ia)^2}\right]$$

$$= \frac{1}{2i} \times \frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

$$\Rightarrow L\{\sin at\} = \frac{a}{s^2+a^2}.$$

$\cos at = \frac{e^{iat}+e^{-iat}}{2}$

$$L\{\cos at\} = L\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$$

$\sinh at = \frac{e^{at}-e^{-at}}{2}$

$\cosh at = \frac{e^{at}+e^{-at}}{2}.$

$$= \frac{1}{2}\left\{\frac{1}{s-ia} + \frac{1}{s+ia}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+\cancel{ia}+s-\cancel{ia}}{s^2+a^2}\right\}$$

$$\Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at}+e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$ Ans.

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Change of Scale

$L\{f(at)\} = ?$

By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

Let $at = u \Rightarrow t = \frac{u}{a}$

$a\,dt = du$

$dt = \frac{1}{a}du$

$\therefore$ When $t = 0$, then $u = 0$,
$t = \infty$, then $u = \infty$.

$$L\{f(u)\} = \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} \cdot f(u)\,du$$

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

→ First shifting Theorem

If Laplace $L\{f(t)\} = F(s)$, $s > 0$

then $L\{e^{at} f(t)\} = F(s-a)$, $s-a > 0$

Proof: By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) = F(s)$$

$$L\{e^{at} \cdot f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)\,dt = \int_0^\infty e^{-t(s-a)} f(t)\,dt$$

$$= F(s-a)$$

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(i) $L\{e^{at}\cos bt\} = ?$

$L\{\cos bt\} = \frac{s}{s^2+b^2}$

By using shifting theorem

$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

(ii) $L\{e^{at}\sin bt\} = ?$

$L\{\sin bt\} = \frac{b}{s^2+b^2}$

By using first shifting

$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

(iii) $L\{e^{at}\cosh bt\} = ?$

$L\{\cosh bt\} = \frac{s}{s^2-b^2}$

$\therefore L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$.

(iv) $L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$.

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→ Laplace Transformation of Derivative

If $L\{f(t)\} = F(s)$

then $L\{f'(t)\} = ?$

Proof: By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = \int_0^\infty \underset{I}{e^{-st}} \underset{II}{f'(t)}\,dt$$

$$= \left[e^{-st} \cdot f(t)\right]_0^\infty + s\int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$= f(0) + sF(s)$$

$$\therefore \quad L\{f'(t)\} = sL\{f(t)\} + f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f''(0).$$

→ Multiplication by 't'

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$$

Example:

$$L\{t\cos t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= (-1)\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{-a^2+s^2}{(s^2+a^2)^2} = \frac{(s+a)(s-a)}{(s^2+a^2)^2}$$

$$L\{t\cos t\} = \frac{s^2-1^2}{(s^2+1^2)^2} \quad \text{Ans.}$$

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Find the laplace transformation;

$$L\{e^t . t^2 . \cos 2t\}$$

$$\therefore \quad L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$L.\{t^2 . \cos 2t\} = \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{-2s(s^2+4)^2 - (4-s^2)\times 2(s^2+4)(2s)}{(s^2+4)^4}$$

$$\therefore \frac{-2s\cancel{(s^2+4)}\left[+(s^2+4)+2(4-s^2)\right]}{(s^2+4)^4}$$

$$= \frac{-2s\left[12-s^2\right]}{(s^2+4)^3}$$

$$\therefore L\{e^t . t^2 . \cos 2t\} = \frac{-2(s-1)\left[12-(s-1)^2\right]}{((s-1)^2+4)^3}$$

$$= \frac{2(s-1)\left[(s-1)^2-12\right]}{((s-1)^2+4)^3}$$

Ans.

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→ Laplace Transformation of Integral.

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

Ex: If Laplace transformation of

$$L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{p^{3/2}}. \text{ Find } L\left\{\frac{1}{\sqrt{\pi t}}\right\}$$

$$\therefore L\{f'(t)\} = sF(s) + f(0)$$

$$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = p.\frac{1}{p^{3/2}} + 0 = \frac{1}{p^{1/2}} \text{ Ans.}$$

→ Find $L\{\sin\sqrt{t}\}$

$$\therefore \sin ax = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} - \frac{(ax)^7}{7!} + \cdots$$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

and

$$L\{\sin\sqrt{t}\} = L\{\sqrt{t}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots$$

$$= \frac{\gamma\frac{1}{2}+1}{s^{3/2}} - \frac{1}{3!}\left(\frac{\gamma\frac{3}{2}+1}{s^{5/2}}\right) + \frac{1}{5!}\left(\frac{\gamma\frac{5}{2}+1}{s^{7/2}}\right)$$

$$= \frac{\frac{1}{2}\times\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\left(\frac{3/2\times\frac{1}{2}\times\sqrt{\pi}}{s^{5/2}}\right) + \frac{1}{5!}\left(\frac{5/2\times 3/2\times 1/2\times\sqrt{\pi}}{s^{7/2}}\right).$$

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$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1-\left(\frac{1}{\cancel{3}\times 2}\times\frac{\cancel{3}}{2s}\right)+\frac{1}{\cancel{5}\times\cancel{3}\times 2}\times\frac{\cancel{5}\times\cancel{3}}{2\times 2\times s^2}\cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1-\frac{1}{4s}+\frac{1}{8s^2}\cdots\right]$$

$e^{ax} =$

$2! \quad 3!$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-\frac{1}{4s}} \quad \underline{\underline{Ans.}}$$

$$e^{-ax} = 1 - ax + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \cdots$$

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} \quad \underline{\underline{Ans.}}$$

Q. Find Laplace transformation of $\sin^3 t$

$\because \sin 3t = 3\sin t - 4\sin^3 t$

$\Rightarrow \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

$$\therefore L\{\sin^3 t\} = \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\}$$

$$= \frac{3}{4}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{4}\left\{\frac{3}{s^2+9}\right\} \quad \underline{\underline{Ans.}}$$

Division by t

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\,ds = \int_s^\infty F(s)\,ds$$

$\underline{\underline{Q.}}$ Find $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

$$\therefore L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$$= \int_s^\infty L\{e^{-at}\}\,ds - \int_s^\infty L\{e^{-bt}\}\,ds$$

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$$\int_s^\infty \frac{1}{s+a} \cdot ds - \int_s^\infty \frac{1}{s+b} ds$$

$$\therefore \left[\ln|s+a|\right]_s^\infty - \left[\ln|s+b|\right]_s^\infty$$

$$\left[\ln\left|\frac{s+a}{s+b}\right|\right]_s^\infty$$

$$\therefore \lim_{s\to\infty} \ln\left|\frac{1+\cancelto{0}{a/s}}{1+b/s}\right| - \ln\left|\frac{s+a}{s+b}\right|$$

$$= -\ln\left|\frac{s+a}{s+b}\right| = \ln\left|\frac{s+b}{s+a}\right| \text{ Ans.}$$

Q. $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L(\cos at) - L(\cos bt)$$

$$= \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$= \left[\frac{1}{2}\ln|s^2+a^2| - \frac{1}{2}\ln|s^2+b^2|\right]_s^\infty$$

$$= -\frac{1}{2}\ln\left|\frac{s^2+a^2}{s^2+b^2}\right| = \frac{1}{2}\ln\left|\frac{s^2+b^2}{s^2+a^2}\right| \text{ Ans.}$$

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Q. Find Laplace transformation of

$$L\{t^2 e^{2t}\cos 4t\}$$

$$= L\{\cos 4t\} = \frac{s}{s^2+16}$$

$$L\{e^{2t}\cos 4t\} = \frac{(s-2)}{(s-2)^2+16}$$

$$L\{t^2 e^{2t}\cos 4t\} = \frac{d^2}{ds^2}\left[\frac{(s-2)}{(s-2)^2+16}\right]$$

$$= \frac{d}{ds}\left[\frac{1((s-2)^2+16) - (s-2)[2(s-2)]}{(s-2)^2+16}\right]$$

$$= \frac{d}{ds}\left[\frac{(s-2)^2+16-2(s-2)^2}{(s-2)^2+16}\right] = \frac{d}{ds}\left[1 - 2\frac{(s-2)^2}{(s-2)^2+16}\right]$$

$$\therefore = -\frac{4(s-2)[(s-2)^2+16] - 2(s(s-2)^2)2(s-2)}{((s-2)^2+16)^2}$$

$$= -\frac{4(s-2)[(s-2)^2+16] - 4(s-2)^3}{((s-2)^2+16)^2}$$

$$= -\frac{4(s-2)((\cancel{s-2})^2+16-(\cancel{s-2})^2)}{((s-2)^2+16)^2}$$

$$= \frac{-64(s-2)}{((s-2)^2+16)^2}.\ \text{Ans.}$$

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→ Inverse Laplace Transformation

$$L[f(t)] = F(s)$$

→ F(s) is known as Laplace transformation of f(t).

→ f(t) is known as Inverse Laplace Transformation of F(s).

$$L^{-1}[F(s)] = f(t)$$

→ Laplace Formula

→ $L[t^n] = \frac{n!}{s^{n+1}}$, $n \in \mathbb{Z}$

→ $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, $n \in$ fraction

→ $L[1] = \frac{1}{s}$.

→ $L[\cos at] = \frac{s}{s^2+a^2}$

→ $L[\sin at] = \frac{a}{s^2+a^2}$

→ $L[\cosh at] = \frac{s}{s^2-a^2}$

→ $L[\sinh at] = \frac{a}{s^2-a^2}$

→ First shifting Theorem

$L[e^{at}.f(t)] = F(s-a)$.

→ Inverse Laplace Formula

→ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$, $n \in \mathbb{Z}$

→ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$, $n \in$ fraction

→ $L^{-1}\left[\frac{1}{s}\right] = 1$.

→ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

→ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

→ $\cosh at = L^{-1}\left[\frac{s}{s^2-a^2}\right]$

→ $\sinh at = L^{-1}\left[\frac{a}{s^2-a^2}\right]$

↳ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$.

→ $L^{-1}(F(s-a)) = e^{at} f(t)$

$f(t) = e^{-at} L^{-1}(F(s-a))$

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→ Multiplication by 't'

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

$(-1)^n t^n f(t) = L^{-1}\left[\frac{d^n}{ds^n} F(s)\right].$

→ Division by 't'.

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L(f(t))\,ds$

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

→ $t . L^{-1}\left[\int_s^\infty F(s)\,ds\right] = f(t).$

→ Laplace of Integral

$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}.$

→ $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

→ Find

$L^{-1}\left[\frac{s+2}{s^2+2^2}\right] = L^{-1}\left[\frac{s}{s^2+2^2} + \frac{2}{s^2+2^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+2^2}\right] + L^{-1}\left[\frac{2}{s^2+2^2}\right]$

$= \cos 2t + \sin 2t$

→ Find

$L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3} L^{-1}\left[\frac{3}{s^2+3^2}\right] = \frac{1}{3} . \sin 3t.$

Q. Find

$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t\,dt}{t}$

$\int_0^\infty e^{-st} f(t)\,dt = L(f(t))$

$= \int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}\,dt$, let $f(t) = \sin\sqrt{3}t$

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$\therefore \quad L\left[\frac{f(t)}{t}\right] = L\left[\frac{\sin\sqrt{3}t}{t}\right]$

$\Rightarrow \quad L\left[\frac{\sin\sqrt{3}t}{t}\right] = \int_s^\infty L(\sin\sqrt{3}t)\, ds$

$= \int_s^\infty \frac{\sqrt{3}}{s^2+3^2} \cdot ds$

$= \frac{\sqrt{3}}{\sqrt{3}} \cdot \left[\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\sqrt{3}}\right)$

$= \cot^{-1}\left(\frac{s}{\sqrt{3}}\right)$

As, $\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-t} \sin\sqrt{3}t\, dt$

$\Rightarrow \quad s = 1.$

$\therefore \quad \int_0^\infty \frac{e^{-t}\sin\sqrt{3}t\, dt}{t} = \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ Ans.

Q. Find the $L^{-1}\left[\frac{s+9}{s^2-4s+5}\right]$

$\Rightarrow \quad L^{-1}\left[\frac{s+9}{(s-2)^2+9}\right] = L^{-1}\left[\frac{s}{(s-2)^2+9} + \frac{9}{(s-2)^2+9}\right]$

$= L^{-1}\left[\frac{s-2+2}{(s-2)^2+9} + \frac{9}{(s-2)^2+9}\right]$

$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + L^{-1}\left[\frac{2}{(s-2)^2+9}\right] + L^{-1}\left[\frac{9}{(s-2)^2+3^2}\right]$

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$e^{2t}\cdot\cos 3t + \frac{2}{3}[e^{2t}\cdot\sin 3t] + 3[e^{2t}\sin 3t]$ Ans.

Q. Find

$$L^{-1}\left[\frac{s}{(s-5)^2+3^2}\right]$$

$$\therefore L^{-1}\left[\frac{(s-5)+5}{(s-5)^2+3^2}\right] = L^{-1}\left[\frac{s-5}{(s-5)^2+3^2}\right] + L^{-1}\left[\frac{5}{(s-5)^2+3^2}\right]$$

$$= e^{5t}\cos 3t + \frac{5}{3}e^{5t}\sin 3t.$$

Q. By using partial fraction, solve:

$$L^{-1}\left[\frac{2s+3}{(s-2)(s^2+2s+5)}\right]$$

$$\therefore \frac{2s+3}{(s-2)(s^2+2s+5)} = \frac{A}{(s-2)} + \frac{Bs+C}{s^2+2s+5}$$

$$2s+3 = A(s^2+2s+5)+(Bs+C)(s-2)$$

$$2s+3 = s^2(A+B)+s(2A-2B+C)+5A-2C$$

$$\Rightarrow A+B=0 \qquad \Rightarrow A = 7/13$$

$$2A-2B+C=2 \qquad B = -7/13$$

$$5A-2C=3 \qquad C = -\frac{2}{13}$$

$$\therefore L^{-1}\left[\frac{2s+3}{(s-2)(s^2+2s+5)}\right] = L^{-1}\left[\frac{7}{13(s-2)} + \frac{-7s+(-2)}{13(s^2+2s+5)}\right]$$

$$= \frac{7}{13}L^{-1}\left[\frac{1}{s-2}\right] + \left(\frac{-1}{13}\right)L^{-1}\left[\frac{7s+2}{s^2+2s+5}\right]$$

$$= \frac{7}{13}e^{2t} - \frac{1}{13}L^{-1}\left[\frac{7s}{(s+1)^2+2^2}\right] - \frac{2}{13}L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{7}{13}e^{2t} - \frac{7}{13}L^{-1}\left[\frac{s+1}{(s+1)^2+}\right] + \frac{7}{26}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{2}{13}L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

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$\frac{7}{13}e^{2t} - \frac{7}{13}e^{-t}\cos 2t + \frac{7}{26}(e^{-t}\sin 2t) - \frac{1}{13}(e^{-t}\sin 2t)$. Ans.

$(s+2)^2$
$= s^2+4+4s$

Q. Find

$$L^{-1}\left[\frac{3s}{(s+2)(s^2+4s+8)}\right]$$

$$\therefore \frac{3s}{(s+2)(s^2+4s+8)} = \frac{A}{(s+2)} + \frac{Bs+C}{s^2+4s+8}$$

$$3s = A(s^2+4s+8) + (Bs+C)(s+2)$$

$$\Rightarrow 3s = s^2(A+B) + s(4A+2B+C) + 8A+2C$$

$$\Rightarrow A+B=0$$
$$4A+2B+C=3$$
$$8A+2C=0$$

$$\therefore A = -3/2$$
$$B = 3/2$$
$$C = 6$$

$$\therefore L^{-1}\left[\frac{3s}{(s+2)(s^2+4s+8)}\right] = L^{-1}\left[-\frac{3}{2(s+2)} + \frac{(\frac{3}{2})s+6}{(s^2+4s+8)}\right]$$

$$= -\frac{3}{2}L^{-1}\left[\frac{1}{(s+2)}\right] + \frac{3}{2}L^{-1}\left[\frac{s}{(s+2)^2+4}\right] + L^{-1}\left[\frac{6}{(s+2)^2+4}\right]$$

$$= -\frac{3}{2}e^{-2t} + \frac{3}{2}L^{-1}\left[\frac{s+2-2}{(s+2)^2+4}\right] + \frac{6}{2}L^{-1}\left[\frac{2}{(s+2)^2+2^2}\right]$$

$$= -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-2t}\cos 2t - \frac{3}{2}e^{-2t}\sin 2t + 3e^{-2t}\sin 2t$$

Ans.

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→ Second Shifting Theorem

If $L[f(t)] = F(s)$

then ;

$$f(t) = \begin{cases} f(t-a), & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\therefore L(f(t)) = e^{-as} F(s)$$

Proof:

$$\therefore L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^\infty e^{-st} \cdot f(t-a)\,dt$$

$$= \cancel{\int_0^a e^{-st} (0)\,dt} + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t)\,dt$$

Let $t - a = u$

$\therefore t = u + a$

$dt = du$

$$\int_0^\infty e^{-s(u+a)} \cdot f(u)\,du = e^{-sa} \int_0^\infty e^{-su} \cdot f(u)\,du$$

$$= e^{-sa} \cdot F(s). \text{ Ans.}$$

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→ Unit Step Function (Heaviside Function)

$$u(t-a) = \begin{cases} u(t), \text{ if } > a \\ 0, \text{ if } t < a \end{cases}$$

$$u(t) = \begin{cases} 1, \text{ if } t > 0, \\ 0, \text{ if } t < 0 \end{cases}$$

→ Find Laplace transformation of the unit step function.

By the definition of Laplace Transformation ;

$$\Rightarrow L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(f(t)) = \int_0^\infty e^{-st} u(t)\,dt$$

$$L(u(t)) = \int_0^a e^{-st} \cancel{u(t)}\,dt + \int_a^\infty e^{-st}(1)\,dt$$

0

$$L(u(t)) = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{}$$

$$\Rightarrow L(u(t)) = \frac{e^{-as}}{}.$$

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Q. Find

$$L(f(t-a)u(t-a)) = ?$$

By the definition of Laplace Transform;

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(f(t-a)u(t-a) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0 e^{-st} f(t-a)\underset{0}{\cancel{u}}(t-a) + \int_a^\infty e^{-st} f(t-a)\underset{1}{\cancel{u}}(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)(t-a)\,dt$$

Let $t-a = u$

$\therefore t = u + a$

$dt = du$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u) \cdot \underset{1}{\cancel{u}}\,du$$

$$= e^{-sa} \cdot L[f(u))$$

$$= e^{-sa} \cdot F(s) \quad \text{Ans.}$$

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Q. $L[(t-1)^2 u(t-1)] = ?$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\therefore L[(t-1)^2 u(t-1)] = e^{-1s} \cdot L[t]$$

$$= e^{-s} \cdot \frac{2}{s^2} \quad \text{Ans.}$$

Q. $L(\sin t\, u(t-\pi)) = ?$

$$L(\sin((t-\pi)+\pi)\, u(t-\pi) = e^{-\pi s} \cdot L[-\sin t]$$

$$= \frac{-e^{-\pi s}}{s^2+1} \quad \text{Ans.}$$

Q. Express the following in heaviside unit function.

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t & 2\pi < t < 3\pi \end{cases}$$

$$H(t) = f_1(t)[u_1(t-a_1) - u_2(t-a_2)] +$$
$$f_2(t)[u_2(t-a_2) - u_3(t-a_3)] +$$
$$f_3(t)[u_3(t-a_3) - u_4(t-a_4)] + \cdots$$
$$f_n(t)[tu(t-a_n)]$$

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$$\therefore \; H(t) = \sin t\,[u_1\, t + u_2(t-\pi)] + \sin 2t\,[u_2(t-\pi) + u_3(t-2\pi)] + \sin 3t\,[u_3(t-3\pi)]$$ Ans.

Q. Express the following function in heaviside unit-step function and find their laplace.

$$f(t) = \begin{cases} t^2 & , 1 < t < 2 \\ 4t & 2 < t < 3 \end{cases}$$

$$H(t) = t^2\,[u_1(t-1) + u_2(t-2)] + 4t\,[u_3(t-3)]$$

$$L(H(t)) = L(t^2 u_1(t-1)) + L(t^2 u_2(t-2)) + L(4\,t\,u_3(t-3))$$

$$= L(((t^2+1)-1)\,u_1(t-1)) + L(((t^2+2)-2)\,u_2(t-2)) + 4\,L(((t+3)-3)\,u_3(t-3))$$

$$= e^{-s} L(t^2+1) + e^{-2s} L(t^2+2) + 4.e^{-3s} L(t+3)$$

$$= e^{-s}\left[\frac{3!}{s^3} + \frac{1}{s}\right] + e^{-2s}\left[\frac{3!}{s^3} + \frac{2}{s}\right] + 4.e^{-3s}\left[\frac{2!}{s^2} + \frac{3}{s}\right]$$

Ans.

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Convolution Theorem

If $L^{-1}[f(t)] = F(s)$

and $L^{-1}[g(t)] = G(s)$

$$L^{-1}[f(t)\,g(t)] = F(s) * G(s)$$

$$\therefore \quad F * G = \int_0^t f(u)\,g(t-u)\,du.$$

Q. By using convolution theorem. Solve

$$L^{-1}\left[\frac{1}{(s^2+2)(s^2+4)}\right].$$

By convolution theorem;

$$\therefore \quad L^{-1} \quad = \quad f(t) * g(t)$$

$$\therefore \quad F(s) = \frac{1}{s^2+2}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]$$

$$f(t) = \frac{1}{\sqrt{2}} L^{-1}\left[\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right]$$

$$f(t) = \frac{\sin\sqrt{2}t}{\sqrt{2}}$$

$$G(s) = \frac{1}{s^2+4}$$

$$\therefore \quad g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$g(t) = \frac{1}{2} L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$g(t) = \frac{\sin 2t}{2}$$

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$\therefore \quad f * g = \int_0 f(u)\, g(t-u)\, du$

$\therefore \quad f(u) = \dfrac{\sin\sqrt{2}u}{\sqrt{2}}$

$g(t-u) = \dfrac{\sin 2(t-u)}{2}$

$$\therefore \quad f * g = \int_0^t \frac{\sin\sqrt{2}u}{\sqrt{2}} \cdot \frac{\sin(2t-2u)}{2}\, du$$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$\therefore \quad f * g = \frac{1}{4\sqrt{2}}\left[\int_0^t \cos(\sqrt{2}u - 2t + 2u) - \cos(\sqrt{2}u + 2t - 2u)\, du\right.$$

$\therefore$ Let $\sqrt{2}u - 2t + 2u = v'$ $\qquad u = 0$, $v' = -2t$

$\therefore \quad \sqrt{2}\,du + 2\,du = dv'$

$du = \dfrac{dv'}{\sqrt{2}+2}$ $\qquad u = t$, $v' = \sqrt{2}t$

and

Let $\sqrt{2}u + 2t - 2u = v''$ $\qquad u = 0$, $v'' = 2t$

$du = \dfrac{dv''}{\sqrt{2}-2}$ $\qquad u = t$, $v'' = \sqrt{2}t$

$$\therefore \quad f * g = \frac{1}{4\sqrt{2}}\left[\int_{-2t}^{\sqrt{2}t} \frac{\cos v'\, dv'}{\sqrt{2}+2} - \int_{2t}^{\sqrt{2}t} \frac{\cos v''\, dv''}{\sqrt{2}-2}\right]$$

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$$f * g = \frac{1}{4\sqrt{2}} \left[\frac{\left[\sin v'\right]_{-2t}}{\sqrt{2}+2} - \frac{\left[\sin v''\right]_{2t}^{\sqrt{2}t}}{\sqrt{2}-2} \right]$$

$$f * g = \frac{1}{4\sqrt{2}} \left[\frac{(\sqrt{2}-2)(\sin\sqrt{2}t + \sin 2t) - (\sqrt{2}+2)(\sin\sqrt{2}t - s}{(\sqrt{2})^2 - (2)^2} \right.$$

$$f * g = \frac{1}{4\sqrt{2}} \left[\frac{2\sqrt{2}\sin 2t - 4\sin\sqrt{2}t}{-2} \right]$$

$$f * g = -\frac{\sin 2t}{4} + \frac{\sin\sqrt{2}t}{2\sqrt{2}}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s^2+2)(s^2+4)}\right] = \frac{\sin\sqrt{2}t}{2\sqrt{2}} - \frac{\sin 2t}{4}$$

Ans.

Q. By using convolution theorem solve;

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right].$$

$$\therefore \quad F(s) = \frac{1}{s+1}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$f(t) = e^{-t}$$

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and

$$G(s) = \frac{1}{s+2}$$

$\therefore$ $g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+2}\right]$

$\Rightarrow$ $g(t) = e^{-2t}$

$\therefore$ $f * g = \int_0^t f(u)\, g(t-u)\, du$

$\because$ $f(u) = e^{-u}$

$g(t-u) = e^{-2(t-u)}$

$\therefore$ $f * g = \int_0^t e^{-u} \cdot e^{-2t} \cdot e^{2u} \cdot du$

$= e^{-2t} \cdot \int_0^t e^{u}\, du$

$= e^{-2t} \left[e^{u} \right]_0^t$

$= e^{-2t} \left[e^{t} - 1 \right]$

$f * g = e^{-t} - e^{-2t}$. Ans.

$\therefore$ $L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$. Ans.

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Q. By using convolution theorem:

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right].$$

$\therefore \quad F(s) = \frac{1}{s}.$

$\Rightarrow \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s}\right]$

$f(t) = 1.$

$\therefore \quad G(s) = \frac{1}{s^2-a^2}.$

$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2-a^2}\right]$

$\therefore \quad g(t) = \frac{1}{a} L^{-1}\left[\frac{a}{s^2-a^2}\right]$

$g(t) = \frac{1}{a} \sinh at.$

$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t 1 \cdot \frac{1}{a} \sinh a(t-u)\, du$

$= -\frac{1}{a}\left[\cosh(t-u)\right]_0^t$

$= -\frac{1}{a}\left[\cosh(0) - \cosh(t)\right]$

$= \frac{1}{a}[\cosh t - 1].$ Ans.

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Q. By using convolution theorem Solve :

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

$$\therefore \quad F(s) = \frac{1}{s(s+1)}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$f(t) = L^{-1}\left[\frac{(s+1)-s}{s(s+1)}\right]$$

$$f(t) = L^{-1}\left[-\frac{1}{s+1} + \frac{1}{s}\right]$$

$$f(t) = 1 - e^{-t}$$

$$G(s) = \frac{1}{s+2}$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+2}\right].$$

$$g(t) = e^{-2t}$$

$$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t (1-e^{-t}) e^{-2(t-u)}\, du = e^{-2t} \int_0^t e^{2u}\, du - e^{-3t} \int_0^t e^{2u}\, du$$

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$$f * g = e^{-2t} \cdot \left[\frac{e^{2u}}{2}\right]_0^t - e^{-3t}\left[\frac{e^{2u}}{2}\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2t}-1}{2}\right] - e^{-3t}\left[\frac{e^{2t}-1}{2}\right]$$

$$f * g = \frac{1-e^{-2t}}{2} - \frac{e^{-t}-e^{-3t}}{2}$$

$$f * g = \frac{1-e^{-t}-e^{-2t}+e^{-3t}}{2} \quad \text{Ans.}$$

***

Assignment : By using convolution theorem, solve

$$L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)(s+4)}\right]$$

$$\therefore F(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$\therefore \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

$$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

$\therefore$ Putting $s = -1$, Putting $s = -2$, Putting $s = -3$

$\therefore 1 = A \times 1 \times 2$ $\quad 1 = -B$ $\quad 1 = 2C$

$\boxed{A = 1/2}$ $\quad \boxed{B = -1}$ $\quad \boxed{C = 1/2}$

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$$\therefore f(t) = L^{-1}[F(s)]$$

$$= L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)}\right]$$

$$= L^{-1}\left[\frac{1/2}{(s+1)} - \frac{1}{s+2} + \frac{1/2}{s+3}\right]$$

$$f(t) = \tfrac{1}{2} e^{-t} - e^{-2t} + \tfrac{1}{2} e^{-3t}$$

$$\therefore G(s) = \frac{1}{s+4}$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+4}\right]$$

$$g(t) = e^{-4t}.$$

$$\therefore f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \left(\frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t}\right) \cdot e^{-4(t-u)}\, du$$

$$= \frac{1}{2} e^{-5t} \int_0^t e^{4u}\, du - e^{-6t} \int_0^t e^{4u}\, du + \frac{1}{2} e^{-7t} \int_0^t e^{4u}\, du$$

$$= \frac{1}{2} e^{-5t}\left[\frac{e^{4t}-1}{4}\right] - e^{-6t}\left[\frac{e^{4t}-1}{4}\right] + \frac{1}{2} e^{-7t}\left[\frac{e^{4t}-1}{4}\right]$$

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$$f*g = \left[\frac{e^{4t}-1}{4}\right]\left[\frac{1}{2}e^{-5t} - e^{-6t} + \frac{1}{2}e^{-7t}\right] \text{ Ans}$$

Assignment: State and prove Convolution Theorem:

Statement: If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(\ )] = g(t)$, ~~then~~

$$L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du = (f*g)(t)$$

Proof:

If $L^{-1}[F(s).G(s) = \int_0^t f(u)\,g(t-u)\,du$

$$\Rightarrow F(s).G(s) = L\left[\int_0^t f(u)\,g(t-u)\,du\right]$$

Using definition of Laplace transform

$$\therefore L(f(t)) = \int_0^\infty e^{-st}.f(t)\,dt$$

$$\Rightarrow \quad = \int_{t=0}^{t=\infty} e^{-st}.\left(\int_{u=0}^{u=t} f(u)\,g(t-u)\,du\right)dt$$

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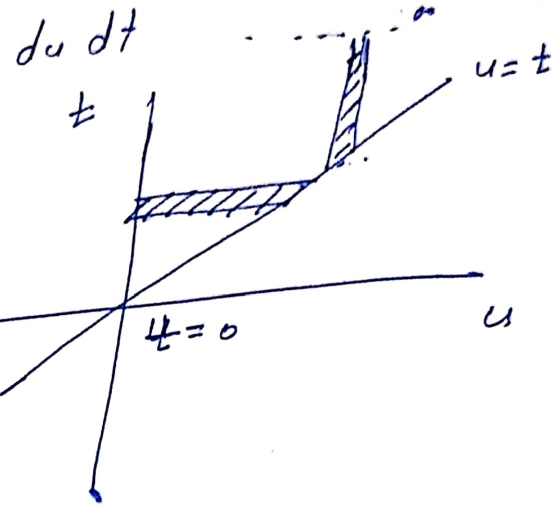
Now, evaluating the integral with the help of change of order of integration.

$$\int_{t=0}^{t=\infty} \int_{u=0}^{u=t} e^{-st} \cdot f(u) g(t-u)\, du\, dt$$



$$\int_{u=0}^{u=\infty} \int_{t=u}^{t=\infty} e^{-st} f(u) g(t-u)\, dt\, du$$

Now, let

$$t - u = z$$

$$dt = dz$$

$$\int_{u=0}^{\infty} \int_{t=u}^{\infty} e^{-st} f(u) g(t-u)\, dt\, du = \int_{u=0}^{\infty} \int_{z=0}^{\infty} e^{-s(u+z)} f(u) g(z)\, dz\, du$$

$$= \int_{u=0}^{\infty} \int_{z=0}^{\infty} e^{-su} f(u) \cdot e^{-sz} g(z)\, dz\, du$$

$$= \left[\int_{u=0}^{\infty} e^{-su} f(u)\, du \right] \times \left[\int_{0}^{\infty} e^{-sz} g(z)\, dz \right]$$

$$= F(s) \cdot G(s).$$ Hence proved.

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LINEAR ALGEBRA

Matrix is an arrangement of a mn elements in rectangular array form where 'm' represent rows and 'n' represent coloumns.

→ Upper Triangular Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be upper triangular if $a_{ij}=0$ for $i>j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

→ Lower Triangular Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be lower triangular if $a_{ij}=0$ for $i<j$.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

→ Nilpotent Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be nilpotent if $A^K=0$, where k is a positive integer.

$$\therefore \quad A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$\Rightarrow \quad A^2 = AA = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} a^2b^2-a^2b^2 & ab^3-ab^3 \\ -a^3b+a^3b & -a^2b^2+a^2b^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K=2.$$

order is 2

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→ Idompotent Matrix

A matrix is said to be idompotent matrix if $A^2 = A$

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

→ Complex Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be complex matrix if elements of the matrix (at least one) is in complex form.

$$A = \begin{bmatrix} 2 & c+id \\ a+ib & 3 \end{bmatrix}$$

→ Complex Conjugate Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be complex conjugate matrix if we occur conjugate of the complex element, i.e

$$a_{ij} = \overline{a_{ij}}$$

→ Hermitian Matrix

A complex matrix is said to be Hermitian matrix if

$$a_{ij} = \overline{a_{ij}}^T.$$

Ex :

$$A = \begin{bmatrix} 2 & a+ib & c+id \\ a-ib & 3 & e+if \\ c-id & e-if & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2 & a-ib & c-id \\ a+ib & 3 & e-if \\ c+id & e+if & 0 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 2 & a+ib & c+id \\ a-ib & 3 & e+if \\ c-id & e-if & 0 \end{bmatrix}$$

$\Rightarrow \quad \bar{A}^T = A \quad ; \quad A \rightarrow$ Hermitian Matrix

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→ Skew Hermitian Matrix

A complex matrix is said to be skew hermitian matrix, if $a_{ij} = -\overline{a_{ij}}^{T}$

NOTE : The principal diagonal of the skew hermitian matrix either 0 or pure complex

Ex :

$$A = \begin{bmatrix} 0 & a+ib & c+id \\ -(a-ib) & 0 & e+if \\ -(c-id) & -(e-if) & 0 \end{bmatrix}$$

$$\therefore \overline{A} = \begin{bmatrix} 0 & a-ib & c-id \\ -(a+ib) & 0 & e-if \\ -(c+id) & -(e+if) & 0 \end{bmatrix} = -\overline{A} = \begin{bmatrix} 0 & -(a-ib) & -(c-id) \\ a+ib & 0 & -(e-if) \\ c+id & e+if & 0 \end{bmatrix}$$

$$\therefore -\overline{A}^{T} = \begin{bmatrix} 0 & a+ib & c+id \\ -(a-ib) & 0 & e+if \\ -(c-id) & -(e-if) & 0 \end{bmatrix} = A.$$

↳ Skew hermitian matrix

→ Rank

Matrix

Singular Matrix

If the determinant of given matrix $A = 0$

$|A| = 0$

Non-singular matrix

If the determinant of given matrix $A \neq 0$

$|A| \neq 0$

Rank : If we calculate the determinant of the matrix and find the value of determinant equal to zero.

$$AX = B$$

$$X = A^{-1}B, \quad A \neq 0$$

If $|A| = 0$ (Rank of Matrix)

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Definition : Highest order of the ~~matrix~~ non-zero minor of the given matrix is called rank of matrix.

$$A = \begin{vmatrix} 1 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{vmatrix}_{3\times3} = 1(6-6)-12(1-1)+3(4-4)=0$$

$$M = \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}_{2\times2} = 2-4=-2 \neq 0.$$

Rank = 2

→ Echelon Form

AB → post multiplication (column) (N)

pre multiplication (row)

1. Entry should be equal to 1. (First element of matrix)
2. First row deal with second and third row } same for column
3. Second row deal with third row

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Step ① : $R_1 \to R_2$ and R_3 → $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$

$R_2 \to R_3$

⇓

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ → May or may not be equal zero.

⇒ No. of non-zero rows = Rank of Matrix

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Q. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 1 \\ 3 & 2 & 1 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Sol. By using elementary transformation

$$\therefore \quad R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$
$$R_4 \to R_4 - 6R_1$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & -8 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & 0 & 0 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_4 \to R_4 \quad R_3$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -11 & 5 \end{bmatrix}$$

Rank of Matrix = No. of non-zero raw = 4.

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Q. Find the ~~mo~~ rank of matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Sol.

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, Rank = 2.

Rank → Echelon Form
- → Row ET → upper Δ mat
- → Column ET → lower Δ mat

diagonal matrix ← $D = I_1 A I_2$

$I_1 \to$ row ET , $I_2 \to$ col ET

Always use equivalent sign ~

$a_{33} \longrightarrow$ zero or non-zero

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→ Linear system of equation

(i) Homogenous $b_1 = b_2 = b_3 = 0$ (c always)

(ii) non-Homogenous $b_1 = b_2 = b_3 \neq 0$ (c or ic)

→ Augmented Matrix

Role: Solution for L.S.E

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

$$A \cdot X = B$$

→ If 'A' is invertible (inverse)

then $X = A^{-1}B$, $|A| \neq 0$

→ If $\det |A| = 0$, then how to calⁿ? → Augmented matrix

Notation: $[A|B]$

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

→ Rank of augmented matrix $= c = R[A|B]$

Note:→ Notation of rank of mat; $r(A)$ or $\rho(A)$

→ Consistent and Inconsistent

Consistent ↓ Solution

You can calculate value of unknown

Inconsistent ↓ No solution

You can't calculate value of unknown

Rank of augmented matrix $[A|B]$ = ROM A

$$\left.\begin{array}{l} R[A|B] = r(A) \\ c = r(A) = \rho(A) \end{array}\right\} c$$

Rank of ag. mat $\neq$ ROM A

$$\left. c \neq r(A) \text{ or } \rho(A) \right\} IC$$

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→ Nature of solution of LSE

Solution → unique
Solution → infinite number of solution

Unique → no. of unknown = R OM

$n = r(A)$

if no. of sol → $n > r(A)$
infinite solution

Q. Solve the system of equations

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = -7$$

Sol.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -7 \end{bmatrix}$$

augmented matrix → $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & -7 \end{array}\right]$

By using echelon form and applying ET

$$R_2 \to R_2 - 3R_1$$
$$R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

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$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 6 & 26 \end{array}\right]$$

Rank of mat = 3

$c = R[A|B] = 3$

$r(A) = 2$ IC system

Solve the linear equations

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21.$$

Given:

System of equations can be expressed as:

$$AX = B$$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

Using echelon form to calculate rank of augmented matrix

$$\therefore \quad [A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

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By using elementary row transformation:

$R_2 \rightarrow R_2 - 4R_1$

$R_3 \rightarrow R_3 - 15R_1$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of augmented matrix $[A|B]$

= No. of non-zerow rows

= 2.

And,

Rank of matrix A = No. of non-zero rows

= 2

Hence, ~~the~~ $R[A|B] = r(A)$ system is consistent

Rank of matrix $A = 2$

No. of unknown = 3

$\therefore$ $n > r$ (Infinite no. of solution)

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→ Eigenvalues and Eigenvectors
(Characteristic values and characteristic vectors)

1. Characteristic Matrix (Eigen Matrix)
2. Characteristic polynomial
3. Characteristic equation
4. Characteristic roots.

Vector
Column vector
Row vector

Eigenvalues
If $A = [a_{ij}]_{m\times n}$ be a square matrix of order 'n', λ is in indeterminant form with identity matrix, then $a - \lambda i$ is called characteristic matrix.

$$\therefore \quad A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we take determinant of the characteristic matrix we arrive at characteristics polynomial.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \det(A - \lambda I)$$

If we equate characteristic polynomial with 0, we obtain characteristic equation.

$$\det(A - \lambda I) = \text{characteristic polynomial} = 0$$

$$\text{Characteristic equation} = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

If we calculate characteristic equation $|A - \lambda I| = 0$, we get the roots of the characteristic equation, obtained roots are called Eigenvalues or characteristic roots or latent roots.

NOTE: The set of eigenvalues is called spectrum.

Eigenvector (Characteristic vector)
Let A be a square matrix,

$$A - \lambda I = 0,$$

if X is any vector (column vector), then the eigen-vector is defined as $(A - \lambda I)X = 0$, where 0 is null matrix.

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Q. Find the eigenvalue or the characteristic value of the matrix :

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}$$

$\therefore$ To determine eigenvalue of the given matrix, 'A'; We will construct characteristic matrix, which is given as;

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}$$

Now, characteristic polynomial;

$\therefore$ $|A - \lambda I| = (5-\lambda)(2-\lambda) = \det(A - \lambda I)$

Characteristic equation is given as:

$|A - \lambda I| = 0$

$\therefore$ $(5-\lambda)(2-\lambda) = 0$

$\Rightarrow$ $\lambda = 5, 2$ $\rightarrow$ These are the eigenvalues of the given matrix.

Now, Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 5$

$\therefore$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

As $\lambda = 5$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\therefore$ $2x_2 = 0$

$0x_1 - 3x_2 = 0$

$\Rightarrow x_1 = k$

$x_2 = 0$

Hence;

$$X_1 = \begin{bmatrix} k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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Eigenvector X_2 corresponding to eigenvalue $X_2 = 2$.

$\therefore \quad (A - \lambda I) X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As, $\lambda = 2$

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 3x_1 + 2x_2 = 0$

$\Rightarrow \quad x_1 = 2$

$x_2 = -3$

Hence

$$X_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \text{Ans.}$$

NOTE: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A. Then, $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are eigenvalues of A^{-1}.

Q. Find the eigenvalue and eigenvector of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

Characteristic Matrix:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$$

Now, characteristic polynomial

$$|A - \lambda I| = (3-\lambda)(2-\lambda) - 2$$
$$= 6 - 3\lambda - 2\lambda + \lambda^2 - 2$$
$$|A - \lambda I| = \lambda^2 - 5\lambda + 4$$

Characteristic equation;

$$|A - \lambda I| = 0$$

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$\therefore \quad \lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 4\lambda - \lambda + 4 = 0$

$\lambda(\lambda - 4) - 1(\lambda - 4) = 0$

$(\lambda - 1)(\lambda - 4) = 0$

Hence, $\lambda = 1, 4 \rightarrow$ These are the eigenvalues of the given matrix

Now, Eigenvector X_1 corresponding to Eigenvalue $\lambda_1 = 1$.

$\Rightarrow \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \lambda = 1$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$x_1 = 1$

$x_2 = -2$

$\therefore \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Ans.

and; Eigenvector X_2 corresponding to Eigenvalue $\lambda_2 = 4$.

$\Rightarrow \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\Rightarrow x_1 = x_2 = k$

$\therefore \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $k \neq 0$. Ans.

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Q. Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}$$

Characteristic Matrix ;

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix}$$

Characteristics polynomial ;

$$|A - \lambda I| = (3-\lambda)^2 - 9 = \lambda^2 - 6\lambda$$

Characteristics equation ;

$$|A - \lambda I| = 0$$

$$\lambda^2 - 6\lambda = 0$$

$\lambda = 0, 6 \longrightarrow$ These are the eigenvalues of the given matrix.

$\therefore$ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 0$.

$\therefore \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As ; $\lambda = 0$

$$\begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$

$\therefore \quad x_1 = 1$

$x_2 = -3$

Hence,

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

and, eigenvector X_2 corresponding to eigenvalue $\lambda_2 = 6$.

$$(A - \lambda I)X = 0$$

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$$\begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \lambda = 6$

$$\begin{bmatrix} -3 & 1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + x_2 = 0$$

$x_1 = 1$
$x_2 = 3$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{Ans.}$$

Q. Find eigenvalue and eigenvector of the matrix

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ Characteristics Matrix :

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}$$

Characteristics Polynomial :

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = (4-\lambda)^2$$

Characteristics equation :

$|A - \lambda I| = 0$

$(4-\lambda)^2 = 0$

$\lambda = 4, 4 \rightarrow$ These are the eigenvalues of the given matrix.

As, eigenvalues are repeated and matrix is symmetrical.
Hence; only one eigenvector exist for ~~this~~ the matrix ;
Eigenvector X corresponding to eigenvalue $\lambda = 4$.

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$$(A - \lambda I)X = 0$$

$$\therefore \begin{bmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0 \cdot x_1 + 0 \cdot x_2 = 0$$

$$\therefore \quad x_1 = x_2 = k, \quad \text{where } k \neq 0.$$

Hence;

$$X = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{Ans.}$$

→ Cayley-Hamilton Theorem

Every square matrix satisfies it's own characteristics equation.

Q. Verify the Cayley-Hamilton theorem for the given matrix:

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$$

Constructing characteristic matrix:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$$

Characteristic polynomial:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 3$$

Characteristic ~~po~~ equation:

$$(5-\lambda)(2-\lambda) - 3 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 7\lambda + 7 = 0$$

Replacing 'λ' with matrix A

$$\therefore \quad A^2 - 7A + 7I = 0$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 25+3 & 5+2 \\ 15+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 28 & 7 \\ 21 & 7 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 28 & 7 \\ 21 & 7 \end{bmatrix} - \begin{bmatrix} 35 & 7 \\ 21 & 14 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Verified.}$$

Hence, matrix A satisfies it's own characteristic equation.

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Find A^{-1}.

$\therefore \quad A^2 + (-7)A + 7I = 0$

Pre-multiplying with A^{-1}.

$$A - 7I + 7A^{-1} = 0$$

$$\therefore \quad A^{-1} = \frac{1}{7}(7I - A)$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad \underline{\text{Ans}}.$$

Q. Verify the Cayley-Hamilton theorem for the matrix:

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 6-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow \quad |A - \lambda I| = (6-\lambda)(1-\lambda) - 12$$
$$= 6 - 6\lambda - \lambda + \lambda^2 - 12$$
$$= \lambda^2 - 7\lambda - 6$$

Characteristics equation:

$$|A - \lambda I| = 0$$
$$\lambda^2 - 7\lambda - 6 = 0$$

Replacing λ with matrix 'A'.

$$A^2 - 7A - 6I = 0$$

$$\therefore \quad A^2 = A.A = \begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36+12 & 18+3 \\ 24+4 & 12+1 \end{bmatrix} = \begin{bmatrix} 48 & 21 \\ 28 & 13 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 48 & 21 \\ 28 & 13 \end{bmatrix} - \begin{bmatrix} 42 & 21 \\ 28 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, matrix 'A' satisfies it's own characteristic equation.

Verified

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Find A^{-1}.

$$A^2 - 7A - 6I = 0$$

$\therefore$ Pre-multiplying by A^{-1}.

$$A - 7I - 6A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{6}(A - 7I).$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 3 \\ 4 & -6 \end{bmatrix}$$ Ans.

→ Diagonilization:

To represent the diagonalization of any matrix A i.e

$$D = P^{-1}AP$$

Condition should be exist, where P is model matrix.

Q. What is model matrix?

A matrix which is construct with the help of the eigenvector is called model matrix.

Ex: Find the diagonalization of matrix A.

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\therefore A - \lambda I = \begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)$$

$$\Rightarrow |A - \lambda I| = 0$$

$$(5-\lambda)(2-\lambda) = 0$$

$\therefore \lambda = 2, 5 \longrightarrow$ Eigenvalues of the given matrix A.

$\therefore$ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 2$

$$\therefore (A - \lambda I)X = 0$$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 2x_2 = 0$$

$$\therefore x_1 = 2$$
$$x_2 = -3$$

$$X_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

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~~Eigenvalue~~

Eigenvector X_2 corresponding to eigenvalue $\lambda_2 = 5$

$\therefore$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$
$$\begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 \cdot x_1 + 2x_2 = 0$$
$$0 \cdot x_1 - 3x_2 = 0$$

$\therefore$ $x_1 = k$
$x_2 = 0$

$$X_2 = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To construct model matrix (P)

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix}$$

$\therefore$ $|P| = 3k$

$$\text{adj}(P) = \begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj}(P)}{|P|} = \frac{1}{3k}\begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix}$$

$$D = P^{-1}AP$$
$$= \frac{1}{3k}\begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix}$$
$$= \frac{1}{3k}\begin{bmatrix} 0 & -2k \\ 15 & 10 \end{bmatrix}\begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix} = \frac{1}{3k}\begin{bmatrix} 6k & 0 \\ 0 & 15k \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{Ans}$$

Principal matrix of D = Eigenvalue of A.

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Find D^8

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \; ; \quad D^8 = \begin{bmatrix} 2^8 & 0 \\ 0 & 5^8 \end{bmatrix}$$ Ans.

Q. Find the diagonalization of matrix :

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$\therefore \quad A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix}$

$$|A - \lambda I| = (3-\lambda)^2 - 4$$
$$= 9 + \lambda^2 - 6\lambda - 4$$
$$= \lambda^2 - 6\lambda + 5$$

$|A - \lambda I| = 0$

$\therefore \quad \lambda^2 - 6\lambda + 5 = 0$

$\lambda^2 - 5\lambda - \lambda + 5 = 0$

$\lambda(\lambda - 5) - 1(\lambda - 5) = 0$

$(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1, 5 \longrightarrow$ Eigenvalues of the given matrix A.

Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 1$

$\therefore \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$x_1 = 1$
$x_2 = -2$

$$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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Eigenv~~alue~~ ector X_2 corresponding to eigenvalue $\lambda_2 = 5$

$$(A - \lambda I)X = 0$$

$$\therefore \begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -2x_1 + x_2 = 0$$

$$x_1 = 1$$

$$x_2 = 2,$$

$$X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ Now;

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$|P| = 2 + 2 = 4$$

$$adj(P) = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore D = P^{-1}AP$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-4 & 2-3 \\ 6+4 & 2+3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 10 & +5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$D = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$ Ans.

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Solution of Differential Equations using Matrix Method.

Q.

$$y_1' = -2y_1 + y_2$$
$$y_2' = y_1 - 2y_2$$

Let us consider;

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

$\therefore$ $$Y' = AY \quad \text{--- (1)}$$

$\Rightarrow$ $$Y' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} Y$$

Let $y = e^{\lambda t}X$ be the solution of (1).

$\therefore$ $$\lambda . e^{\lambda t}X = A . e^{\lambda t}X$$

$\therefore$ $$\lambda X = AX$$

Hence,

$$(A - \lambda I)X = 0$$

Calculating eigenvalues ;

$$\det(A - \lambda I) = 0$$

$\therefore$ $$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)^2 - 1 = 0$$
$$4 + \lambda^2 + 4\lambda - 1 = 0$$
$$\lambda^2 + 4\lambda + 3 = 0$$
$$\lambda^2 + 3\lambda + \lambda + 3 = 0$$

$$\lambda(\lambda+3) + 1(\lambda+3) = 0$$
$$(\lambda+1)(\lambda+3) = 0$$
$$\lambda = -3 \text{ and } -1.$$

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$\therefore$ Eigenvector. X_1 corresponding to eigenvalue $\lambda_1 = -3$.

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} X = 0$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 = 0$$

$$\Rightarrow x_1 = 1 \text{ and } x_2 = -1.$$

Hence;

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

and, Eigenvector X_2 corresponding to eigenvalue $\lambda_2 = -1$.

$$\Rightarrow (A - \lambda I)X = 0$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = 1 \text{ and } x_2 = 1$$

Hence,

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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Therefore,

Solution of the given differential equation;

$$y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$y = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad y_1 = e^{-3t} + e^{-t}$

$\quad\quad y_2 = -e^{-3t} + e^{-t}$. Ans.

Q. $y_1' = 5y_1 + 22y_2$

$y_2' = y_1 + 2y_2$

$\therefore$ Consider

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and $$A = \begin{bmatrix} 5 & 22 \\ 1 & 2 \end{bmatrix}$$

$\therefore \quad (A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} X = 0$$

$\therefore$ Eigenvalues are given as;

$\det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(5-\lambda)(2-\lambda) - 22 = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 - 22 = 0$

$\lambda^2 - 7\lambda - 12 = 0$

$$\therefore \lambda = \frac{7 \pm \sqrt{97}}{2}$$

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$\therefore$ Eigenvector X_1 corresponding to eigenvalues $\lambda_1 = \frac{7}{2} + \frac{\sqrt{97}}{2}$

$\therefore$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{97}}{2} & 22 \\ 1 & -\frac{3}{2} - \frac{\sqrt{97}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\frac{3}{2} - \frac{\sqrt{97}}{2}\right)x_1 + 22x_2 = 0$$

$\therefore$ $x_1 = 22$ and $x_2 = -\frac{3}{2} + \frac{\sqrt{97}}{2}$

Hence, $X_1 = \begin{bmatrix} 22 \\ -\frac{3}{2} + \frac{\sqrt{97}}{2} \end{bmatrix}$

and eigenvector X_2 corresponding to eigenvalues $\lambda_2 = \frac{7}{2} - \frac{\sqrt{97}}{2}$.

$\therefore$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{97}}{2} & 22 \\ 1 & -\frac{3}{2} + \frac{\sqrt{97}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \left(\frac{3}{2} + \frac{\sqrt{97}}{2}\right)x_1 + 22x_2 = 0$$

$x_1 = 22$ and $x_2 = -\frac{3}{2} - \frac{\sqrt{97}}{2}$.

Hence $X_2 = \begin{bmatrix} 22 \\ -3/2 - \frac{\sqrt{97}}{2} \end{bmatrix}$

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Therefore,

Solution to the given differential equations ;

$$y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$\therefore \quad y = e^{\left(\frac{7+\sqrt{97}}{2}\right)t} \begin{bmatrix} 22 \\ -3/2 + \frac{\sqrt{97}}{2} \end{bmatrix} + e^{\left(\frac{7-\sqrt{97}}{2}\right)t} \begin{bmatrix} 22 \\ -3/2 - \sqrt{97}/2 \end{bmatrix}$$

$$y_1 = 22\, e^{\left(\frac{7+\sqrt{97}}{2}\right)t} + 22\, e^{\left(\frac{7-\sqrt{17}}{2}\right)t}$$

$$y_2 = \left(-\frac{3}{2} + \frac{\sqrt{97}}{2}\right) e^{\left(\frac{7+\sqrt{97}}{2}\right)t} + \left(-\frac{3}{2} - \frac{\sqrt{97}}{2}\right) e^{\left(\frac{7-\sqrt{97}}{2}\right)t} \quad \text{Ans.}$$

Q. Solve the following differential equations using matrix method.

$$y_1' = 4y_1 + 3y_2$$
$$y_2' = 2y_1 + y_2$$

$$\therefore \quad A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Consider, $Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\therefore \quad (A - \lambda I)X = 0$$

For eigenvalues ;

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 6 = 0$$
$$4 - 4\lambda - \lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 5\lambda - 2 = 0$$
$$\lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5}{2} \pm \frac{\sqrt{33}}{2}$$

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$\therefore$ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = \frac{5}{2} + \frac{\sqrt{33}}{2}$.

$\therefore$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & 3 \\ 2 & -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\left(\frac{3}{2} - \frac{\sqrt{33}}{2}\right)x_1 + 3x_2 = 0$

$x_1 = 3$ and $x_2 = -\frac{3}{2} + \frac{\sqrt{33}}{2}$

Hence, $X_1 = \begin{bmatrix} 3 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}$

and eigenvector X_2 corresponding to eigenvalue $\lambda_2 = \frac{5}{2} - \frac{\sqrt{33}}{2}$.

$(A - \lambda I)X = 0$

$$\begin{bmatrix} \frac{3}{2} + \frac{\sqrt{33}}{2} & 3 \\ 2 & -\frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\left(\frac{3}{2} + \frac{\sqrt{33}}{2}\right)x_1 + 3x_2 = 0$

$\therefore$ $x_1 = 3$ and $x_2 = -3/2 - \frac{\sqrt{33}}{2}$

Hence, $X_1 = \begin{bmatrix} 3 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$

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Therefore,

Solution of the given differential equations;

$$y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$y = e^{\left(\frac{5+\sqrt{33}}{2}\right)t} \begin{bmatrix} 3 \\ -\frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix} + e^{\left(\frac{5-\sqrt{33}}{2}\right)t} \begin{bmatrix} 3 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$$

$$y_1 = 3e^{\left(\frac{5+\sqrt{33}}{2}\right)t} + 3e^{\left(\frac{5-\sqrt{33}}{2}\right)t}$$

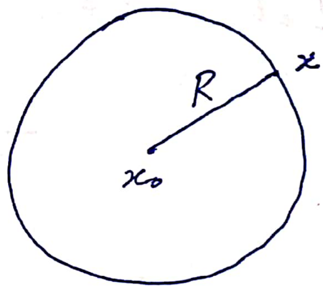
$$y_2 = \left(-\frac{3}{2} + \frac{\sqrt{33}}{2}\right)e^{\left(\frac{5+\sqrt{33}}{2}\right)t} + \left(-\frac{3}{2} - \frac{\sqrt{33}}{2}\right)e^{\left(\frac{5-\sqrt{33}}{2}\right)t}$$ Ans.

Power Series

An expression;

$$\sum_{n=0}^{\infty} (x - x_0)^n Z^n$$ is known as Power Series,

where x is the center of of the circle. $(z \in C)$



→ $|x - x_0| = R$
equation of circle.

→ $|x - x_0| < R$
Convergence

→ $|x - x_0| > R$
Divergence.

R is called radius of convergence.

To calculate radius of convergence;
We use;
① Cauchy n^{th} root test
② Ratio Test.

→ Application of the power series in differential equations.
To find solution of differential equation in series form.

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Q. To find the series solution by using Fibonic method of the differential equation of;

$$2x^2y'' + xy' - (x^2+1)y = 0 \quad \text{— ①}$$

Given:

Let $y = \sum_{m=0}^{\infty} C_m x^{m+r}$ is series solution of the given differential equation.

Regular Singular Point.

$$2(x-x_0)^2 y'' + (x-x_0)y' - (x^2+1)y = 0$$

$$y = \sum_{m=0}^{\infty} a_m (x-x_0)^{m+r}, \quad x_0 = 0 \rightarrow \text{Regular singular point.}$$

$$A_0(x)y'' + A_1(x)y' + A_2(x)y = 0$$

Let $x = x_0$ is regular singular point.

$$\frac{p_1}{x-x_0} = \frac{A_1}{A_0}$$

$$\therefore \quad A_1 = \frac{p_1 A_0}{x-x_0}, \quad x \neq x_0 .$$

$$y = \sum_{m=0}^{\infty} C_m x^{m+r} \quad \text{— ②}$$

Differentiate eq (2) w.r.t x

$$y' = \sum_{m=0}^{\infty} C_m (m+r) x^{m+r-1} \quad \text{— ③}$$

Differentiating again

$$y'' = \sum_{m=0}^{\infty} C_m (m+r)(m+r-1) x^{m+r-2} \quad \text{— ④}$$

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From eq ①, ②, ③ and ④

$$2\left(\sum_{m=0}^{\infty} C_m (m+r)(m+r-1)x^{m+r-2}\right)x^2 +$$

$$\left(\sum_{m=0}^{\infty} C_m (m+r)x^{m+r-1}\right)(x) - (x^2+1)\left(\sum_{m=0}^{\infty} C_m x^{m+r}\right) = 0$$

$$\left[2\left(\sum_{m=0}^{\infty} C_m (m+r)(m+r-1)\right) + \left(\sum_{m=0}^{\infty} C_m (m+r)\right) - \sum_{m=0}^{\infty} C_m\right]x^{m+r}$$

$$-\left[\sum_{m=0}^{\infty} C_m x^{m+r+2}\right] = 0$$

Indicial equation $(m=0)$ [Lowest power term]

$$2C_0(0+r)(0+r-1)x^{0+r} + C_0(0+r)x^{0+r} - C_0x^{0+r+2} -$$
$$C_0x^{0+r} = 0$$

$\therefore$ $2C_0(r)(r-1)x^r + C_0(r)x^r - C_0x^{r+2} - C_0x^r = 0$

$\therefore$ $C_0\left[2r(r-1) + (r) - \quad\right]x^r - C_0x^{r+2} = 0$

Lowest power term $\Rightarrow x^r$

$\Rightarrow$ $C_0\left[2r(r-1) + (r) - 1\right] = 0$

$2r(r-1) + 1(r-1) = 0$

$(r-1)(2r+1) = 0$

$r = 1$ and $-1/2$.

$r_1 - r_2 = 1 - (-1/2) = 3/2 \neq 0$

Roots are different and not differ by an integer.

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Now, $m=1$.

$$\left[2(C_1(r+1)(r)x^{r+1}) + C_1(r+1)x^{r+1} - C_1x^{r+1}\right] - C_1x^{r+3} = 0$$

$$\therefore \quad C_1\left\{2(r+1)r + 1(r+1) - 1\right\}x^{r+1} - C_1x^{r+3} = 0$$

Lowest power term $\Rightarrow x^{r+1}$

$$\therefore \quad C_1\left\{2r(r+1) + (r+1) - 1\right\} = 0$$

$$C_1\left\{2r(r+1) + (r+1) - 1\right\} = 0$$

$$C_1\left\{2r^2 + 3r\right\} = 0$$

As, $2r^2 + 3r \neq 0$ for $r = 1$ and $-1/2$

$$\Rightarrow \boxed{C_1 = 0}$$

Hence; $C_3 = C_5 = C_7 = \dots = C_{2n+1} = 0$.

Now, $m=2$.

$$\therefore \quad \sum_{m=2}^{\infty} C_m\left\{2(m+r)(m+r-1) + (m+r) - 1\right\}x^{m+r} - \sum_{m=0}^{\infty} C_m x^{m+r+2} = 0$$

$\longrightarrow$ Replacing $m \rightarrow m-2$

$$\therefore \quad \sum_{m=2}^{\infty} C_m\left\{2(m+r)(m+r-1) + (m+r) - 1\right\}x^{m+r} - \sum_{m-2=0}^{\infty} C_{m-2}\, x^{m+r+\cancel{2}-\cancel{2}} = 0$$

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$\therefore\; C_m\{2(m+r)(m+r-1)+(m+r)-1\}x^{m+r} - C_{m-2}x^{m+r} = 0$

Recurrence relation

$\therefore\; C_m = \dfrac{C_{m-2}}{(2(m+r)(m+r-1)+(m+r)-1)}$ $\qquad 3/2 - 1$

$\therefore\; m = 2$

$C_2 = \dfrac{C_0}{2(2+r)(1+r)+(2+r)-1}$

$\begin{array}{r} 44 \\ 14 \\ \hline 176 \\ 44\times \\ \hline 616 \end{array}$

For $r = 1$

$C_2 = \dfrac{C_0}{2(3)(2)+(3)-1} = \dfrac{C_0}{12+2} = \dfrac{C_0}{14}$

For $r = -1/2$

$C_2 = \dfrac{C_0}{2(2-1/2)(1/2)+(2-1/2)-1} = \dfrac{C_0}{3(1/2)+1/2} = \dfrac{C_0}{2}.$

$m = 4$

$C_4 = \dfrac{C_2}{2(4+r)(3+r)+(4+r)-1}$

For $r = 1$

$C_4 = \dfrac{C_2}{2(5)(4)+(5)-1} = \dfrac{C_0}{14(40+4)} = \dfrac{C_0}{14\times 44} = \dfrac{C_0}{616}$

For $r = -1/2$

$C_4 = \dfrac{C_2}{2(4-1/2)(3-1/2)+(4-1/2)-1} = \dfrac{C_2}{2(7/2)(5/2)+(7/2)-1}$

$= \dfrac{C_2}{7(5/2)+5/2} = \dfrac{C_2}{40/2} = \dfrac{C_2}{20} = \dfrac{C_0}{40}.$

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$\therefore$ For $r = 1$

$$y = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5 + \dots$$

$$y = C_0 x + 0x^2 + \frac{C_0}{14} x^3 + 0.x^4 + \frac{C_0}{616} x^5 + \dots$$

$$\Rightarrow \quad y = C_0 \left[x + \frac{x^3}{14} + \frac{x^5}{616} + \dots \right] \text{ Ans.}$$

and For $r = -1/2$

$$y = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5 + \dots$$

$$y = C_0 x + 0x^2 + \frac{C_0}{2} x^3 + 0x^4 + \frac{C_0}{40} x^5 + \dots$$

$$y = C_0 \left[x + \frac{x^3}{2} + \frac{x^5}{40} + \dots \right] \text{ Ans.}$$

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Vector Space and Linear Transformation

→ Vector space: Let V be a non-empty set and elements of V can be matrix, vector, function, etc. If v is an element of V such that $v \in V$, then v is called vector.

Properties of vector space for addition

1. Closure property:
$$a+b=c, \quad a,b,c \in V$$
2. Commutative property:
$$a+b=b+a$$
3. Associative property:
$$(a+b)+c = a+(b+c)$$
4. Existence of unique zero, which belong in V.
$$a+0=0+a=a$$
5. Existence of negative
$$a+(-a)=0.$$

Multiplication Property. closure property

1. $\alpha A = b$, where α is scalar and $a \in V$
2. Left distribution law
$$(\alpha+\beta)a = \alpha a + \beta a$$
3. Right distribution law
$$a(\alpha+\beta) = \alpha a + \beta a$$
4. Existence of the identity element
$$a * 1 = a.$$
5. Associative Law
$$(\alpha\beta)a = \alpha(\beta a)$$

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→ Linear Transformation

$$T : A \to B$$

$$T : \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$T(a_{11}) \to b_{11}$$

Let A and B be two non-empty set such that

$$T: A \to B.$$

where, T is called linear transformation from A to B. If it follows following properties.

(i) If 'α' is scalar and v in V. Then;

$$T(\alpha v) = \alpha T(v)$$

(ii) $T(v_1 + v_2) = T(v_1) + T(v_2)$

(iii) $T(\alpha_1 v_1 + \alpha_2 v_2) = T(\alpha_1 v_1) + T(\alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$.

Example; If $T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$T\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Find

$$T\begin{bmatrix} 4 & 5 \\ 3 & 8 \end{bmatrix} = ?$$

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These handwritten notes are of MTH-S102 taught to us by Prof. D.K. Singh, compiled and organized chapter-wise to help our juniors. We hope they make your prep a bit easier.

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