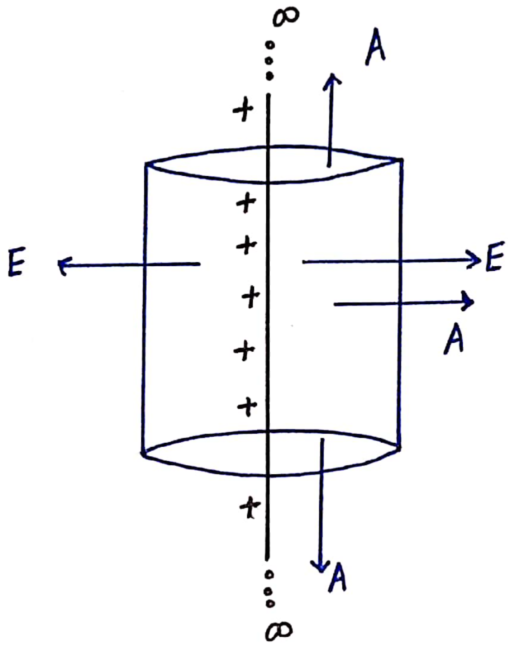
Q. Define the applications of Gauss's Law

(i) Electric Field due to an infinite line of charge.



Let us consider an infinite line of charge, whose linear charge density is given by λ.

∴ According to Gauss's Law

$$\Rightarrow \phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\therefore \phi_E = \oint E\, dA \cos 0^\circ = \frac{Q}{\epsilon_0}$$

$$\phi_E = E \oint dA$$

$$\phi_E = E \cdot 2\pi r l$$

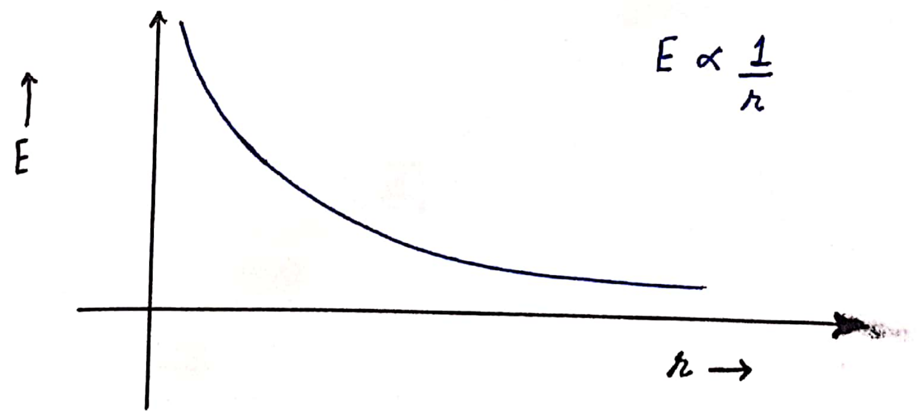
and $\lambda = \frac{Q}{l}$

hence ; $Q = \lambda l$

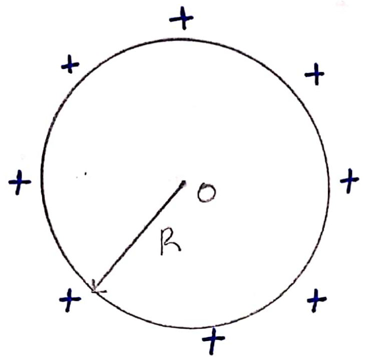
$$\therefore \quad E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi r \epsilon_0} = \frac{2K\lambda}{r}}$$

The variation of electric field with distance can be plotted as;

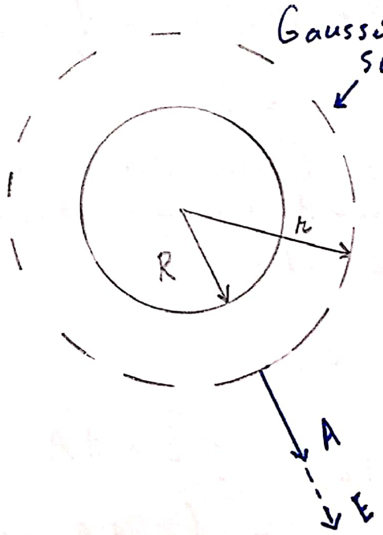


ii) Electric Field due to charged spherical shell.



Let us consider a positively charged spherical shell of radius 'R', with charge Q.
As, shell is hollow inside, all charge resides on it's surface.

Case 1: Electric Field on an external field ($r > R$)



$$\therefore \quad \phi_E = \oint \vec{E} \cdot d\vec{A}$$

As, electric field and area vector are in the same direction

$$\Rightarrow \quad \theta = 0^\circ$$

$$\therefore \quad \phi_E = \oint E\, dA \cos 0^\circ = \oint E\, dA$$

$$= E \oint dA$$

$$\phi_E = E \times 4\pi r^2$$

According to Gauss's Law

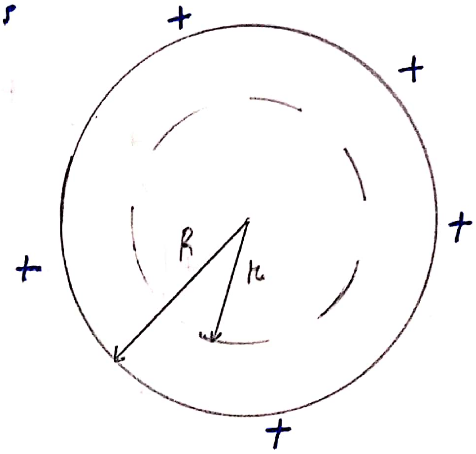
$$\phi_E = \oint E \cdot dA = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad E \times 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi r^2 \epsilon_0}}$$

Case 2: Internal Point ($r < R$)

∴ As, all the charge resides on the surface of spherical shell.



$\Rightarrow \quad Q_{inside} = 0$

Therefore,
according to Gauss's Law:

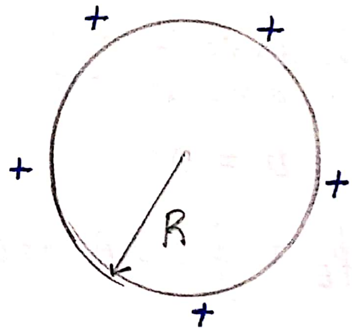
$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

But, as; $Q = 0$

$$\Rightarrow \quad \boxed{E = 0}$$

Case 3: On surface ($r = R$)

As, $r = R$.
The Gaussian surface would be formed overlapping the shell.



Hence,

$$\phi_E = \oint \vec{E} \cdot d\vec{A}$$

$$= \oint E\, dA \cos 0^\circ \quad (\vec{E} \text{ and } \vec{A} \text{ are in the same direction})$$

$$\phi_E = \oint E\, dA$$

$$\phi_E = E \oint dA$$

$$\phi_E = E \cdot 4\pi R^2$$

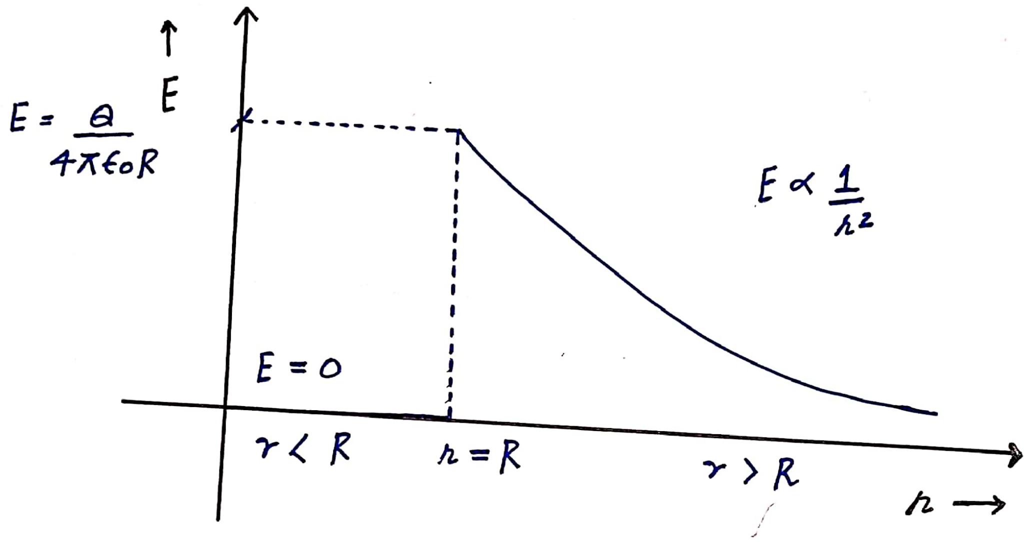
According to Gauss's Law;

$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$E \cdot 4\pi R^2 = Q/\epsilon_0$$

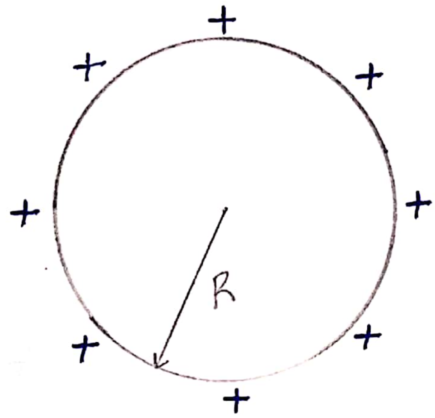
$$\boxed{E = \frac{Q}{4\pi\epsilon_0 R^2}}$$

The variation of electric field with distance can be plotted as;



(iii) Electric Field due to uniformly charged solid sphere.

Case 1: Conducting Sphere



Let us consider a uniformly charged 'conducting' solid sphere with charge Q and radius R.

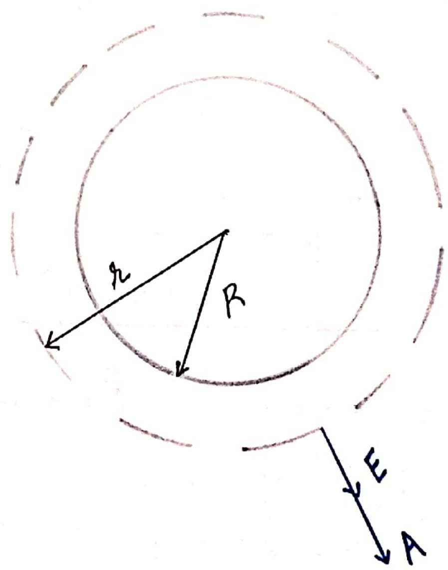
As, the given solid sphere is conducting, therefore all charge will reside on the surface of sphere.

Case 1: External Point $(r > R)$

$\therefore \quad \phi_E = \oint \vec{E} \cdot d\vec{A}$

As, electric field and area vector are in same direction.

$\Rightarrow \quad \theta = 0^\circ$



$\phi_E = \oint E \, dA \cos 0^\circ$

$\phi_E = \oint E \, dA$

$\phi_E = E \oint dA$

$\phi_E = E \times 4\pi r^2$

According to Gauss's Law;

$\phi_E = \oint E \cdot dA = \frac{Q}{\epsilon_0}$

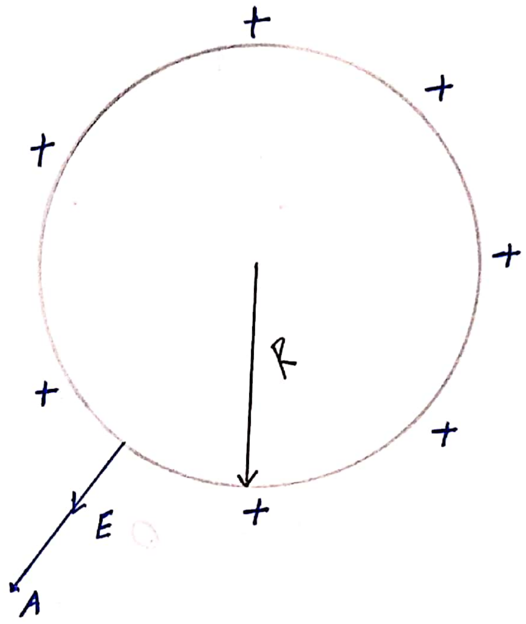
$E \times 4\pi r^2 = \frac{Q}{\epsilon_0}$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$$

Case 2: On surface ($r = R$)

As, $r = R$

The Gaussian surface would be overlapping the sphere.



$$\phi_E = \oint \vec{E} \cdot d\vec{A}$$

As, $\vec{E}$ and $\vec{A}$ are in the same direction.

$$\Rightarrow \theta = 0°$$

$$\therefore \phi_E = \oint E\, dA \cos 0°$$

$$\phi_E = \oint E\, dA$$

$$\phi_E = E \oint dA$$

$$\phi_E = E \times 4\pi R^2$$

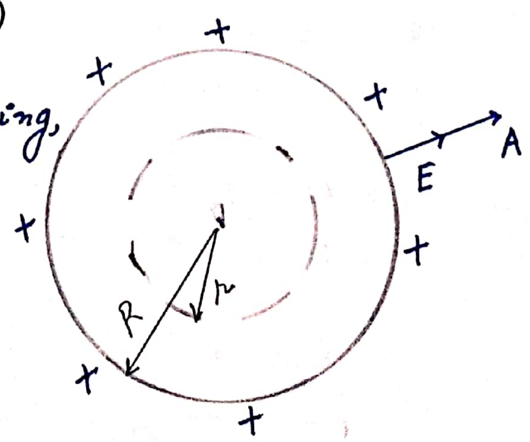
According to Gauss's Law:

$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$E \cdot 4\pi R^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 R^2}}$$

Case 3: Internal Point ($r < R$)



As, the given sphere is conducting, all charge resides on surface.

$$\Rightarrow Q_{inside} = 0$$

According to Gauss's Law

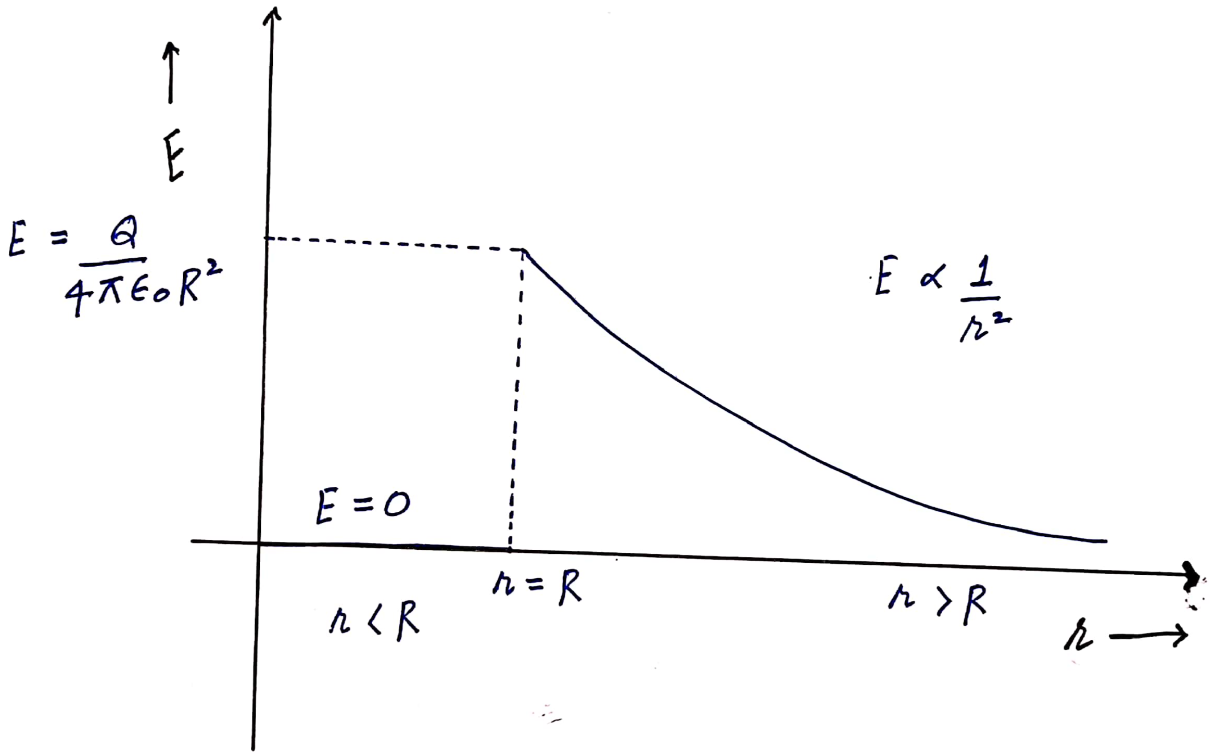
$$\therefore \phi_E = \oint E \cdot dA = \frac{Q}{\epsilon_0}$$

But, as $Q = 0$

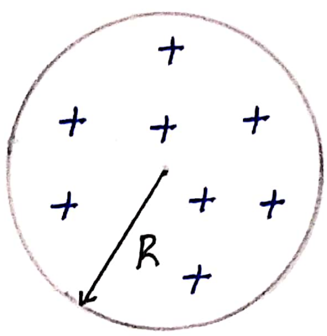
Hence,

$$\boxed{E = 0}$$

The variation of Electric Field with distance (r) can be plotted as ;



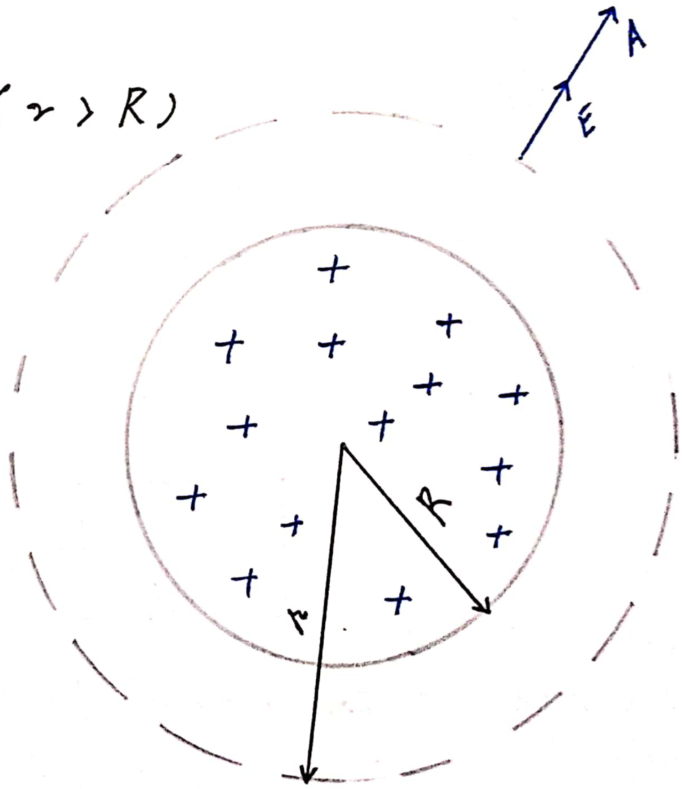
Case 2: Non-Conducting Sphere



Let us consider a uniformly charged solid 'non-conducting' sphere with charge Q, radius R and volume charge density ρ.

$$\therefore \quad \rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{Q}{V}$$

Case 1: External Point ($r > R$)



$$\phi_E = \oint \vec{E} \cdot d\vec{A}$$

As, $\vec{E}$ and $\vec{A}$ both are in the same direction

$$\Rightarrow \quad \theta = 0^\circ$$

$$\phi_E = \oint E\, dA \cos 0^\circ$$

$$= \oint E\, dA$$

$$= E \oint dA$$

$$\phi_E = E \times 4\pi r^2$$

According to Gauss's Law

$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad E \times 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$$

Case 2: On surface $(r = R)$

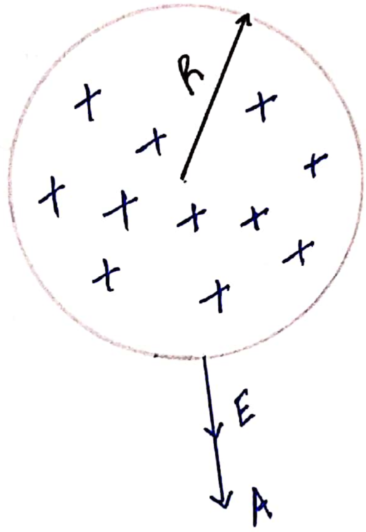
As, $r = R$.

Therefore, the Gaussian surface will overlap the sphere.

$\therefore \quad \phi_E = \oint \vec{E} \cdot d\vec{A}$

As, $\vec{E}$ and $\vec{A}$ are in the same direction.

$\Rightarrow \quad \theta = 0^\circ$



$$\phi_E = \oint E\, dA \cos 0^\circ$$

$$\phi_E = \oint E\, dA$$

$$\phi_E = E \oint dA$$

$$\phi_E = E \times 4\pi R^2$$

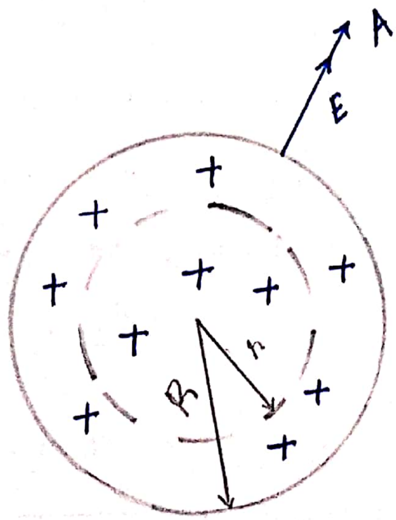
According to Gauss's Law:

$$\phi_E = \oint E \cdot dA = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E \times 4\pi R^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 R^2}}$$

Case 3: Internal Point $(r < R)$



$\therefore \quad \phi_E = \oint \vec{E} \cdot d\vec{A}$

As, $\vec{E}$ and $\vec{A}$ are in the same direction.

$\Rightarrow \quad \theta = 0^\circ$

$$\phi_E = \oint E\, dA \cos 0^\circ$$

$$\phi_E = E \oint dA$$
$$\phi_E = E \times 4\pi r^2$$

According to Gauss's Law:

$$\phi_E = E \times 4\pi r^2 = \frac{Q}{\epsilon_0} \qquad \left[\rho = \frac{Q}{V} \right]$$

$$\Rightarrow E \times 4\pi r^2 = \frac{\rho \times V}{\epsilon_0}$$

$$E \times 4\pi r^2 = \frac{\rho \times 4\pi r^3}{3\epsilon_0}$$

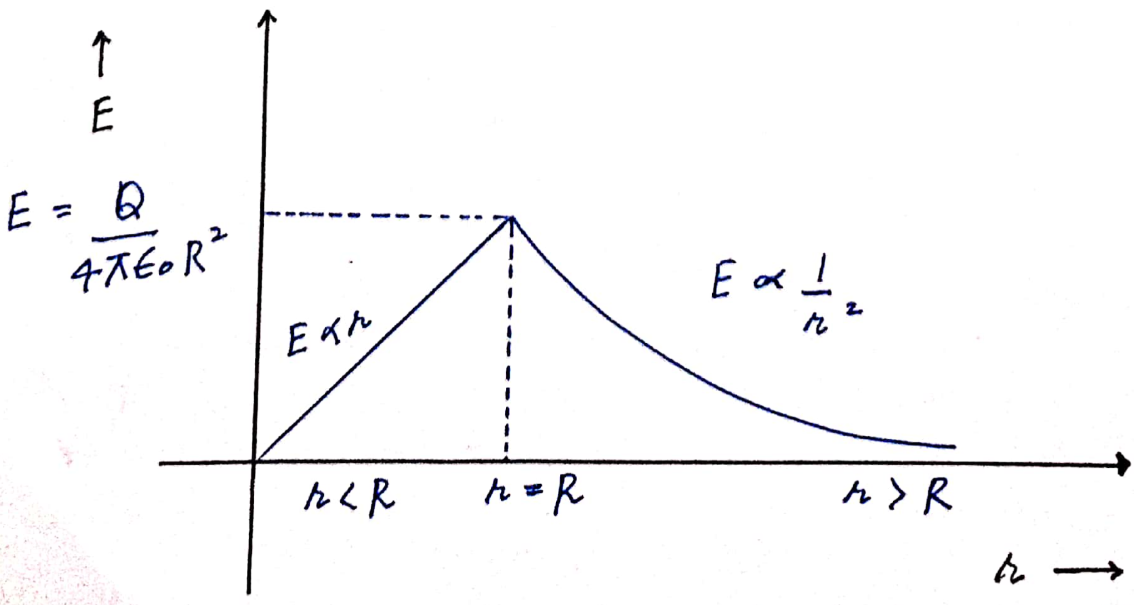
$$E = \frac{\rho r}{3\epsilon_0}$$

Writing $\vec{E}$ in terms of Q

$$E = \frac{Q r \times 3}{4 \times \pi \times \epsilon_0 \times 3 \times R^3}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0} \cdot \frac{r}{R^3}}$$

The variation of Electric field ($\vec{E}$) with distance (r) can be plotted as:

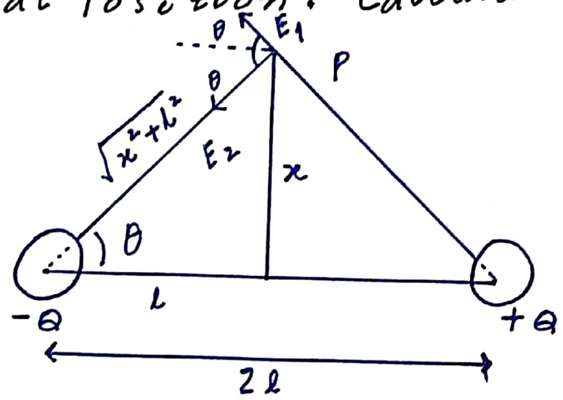


Q. Find the electric field intensity as well as potential of a dipole at:

(i) Equitorial position

(ii) Axial position

(i) Equitorial Position: Calculating Electric field and potentia



Let us consider a dipole comprising of charges -Q and +Q seperated by a distance of 2l.

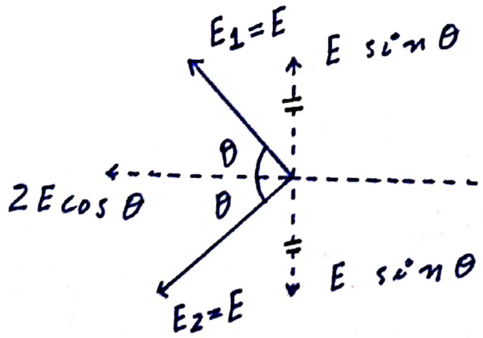
Calculating $\vec{E}$ due to dipole at point P (equitorial) 'x' distance above the axis line

$$\therefore \quad E_1 = \frac{KQ}{(\sqrt{l^2+x^2})^2} = \frac{KQ}{l^2+x^2}$$

and $$E_2 = \frac{KQ}{(\sqrt{l^2+x^2})^2} = \frac{KQ}{l^2+x^2}$$

Hence, $E_1 = E_2 = E$

$\therefore$



$$\Rightarrow \quad E_{net} = 2E\cos\theta$$
$$= 2 \times \frac{KQ}{l^2+x^2} \times \frac{l}{(l^2+x^2)^{1/2}}$$

$$E = \frac{2KQl}{(l^2+x^2)^{3/2}}$$

As ; $\vec{p} = Q \times (2l)$

where, $\vec{p}$ is dipole moment

Q is magnitude of charge

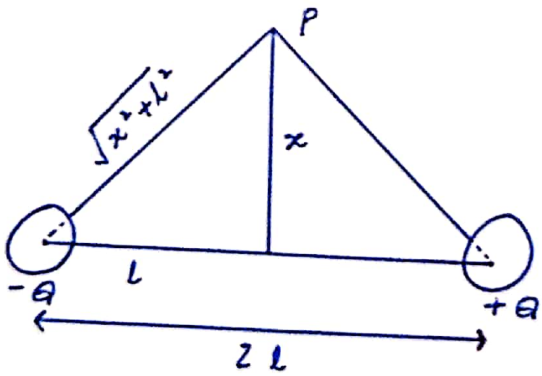
$2l$ is seperation between charges

Hence.

$$\vec{E} = \frac{K\vec{p}}{(x^2+l^2)^{3/2}}$$

As ; $x >>> l$

$$\boxed{\vec{E}_{equitorial} = \frac{K\vec{p}}{x^3}.}$$



Again, considering the same setup and calculating potential at point P (equitorial)

$\therefore$ Potential due to $-Q$ $= V_1 = \frac{-KQ}{(x^2+l^2)^{1/2}}$

Potential due to $+Q$ $= V_2 = \frac{+KQ}{(x^2+l^2)^{1/2}}$

Hence

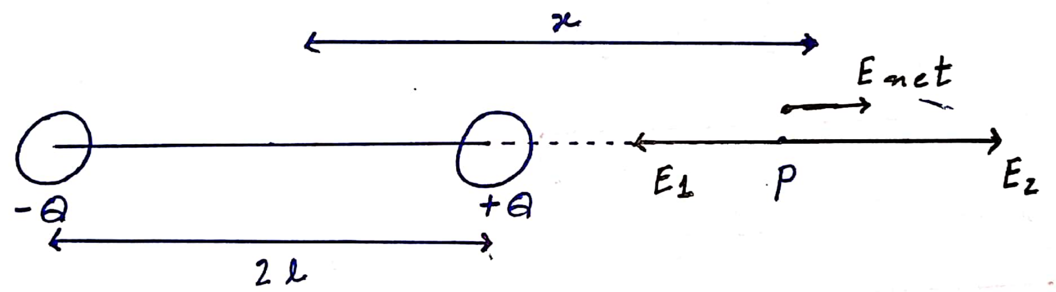
$$V_p = V_1 + V_2$$

$$= \frac{-KQ}{(x^2+l^2)^{1/2}} + \frac{KQ}{(x^2+l^2)^{1/2}}$$

$$\boxed{V_p = 0}$$

This implies, that an equipotential surface maybe assummed for equitorial position of a dipole.

ii) Axial position: Calculating Electric field and potential.



Let us consider a dipole comprising of charges +Q and -Q seperated by distance '2l'.
Assuming a point P lying on the axial line of dipole (axial position) and its distance 'x' from the centre of dipole.

$\vec{E}$ due to $-Q$ $= E_1 = \frac{KQ}{(l+x)^2}$

$\vec{E}$ due to $+Q$ $= E_2 = \frac{KQ}{(x-l)^2}$

Hence,

$$E_{net} = \vec{E_2} - \vec{E_1}$$

$$E_{net} = \frac{KQ}{(x-l)^2} - \frac{KQ}{(x+l)^2}$$

$$= KQ\left[\frac{(x+l)^2 - (x-l)^2}{(x-l)^2(x+l)^2}\right]$$

$$= KQ\left[\frac{4xl}{(x^2-l^2)^2}\right]$$

As,

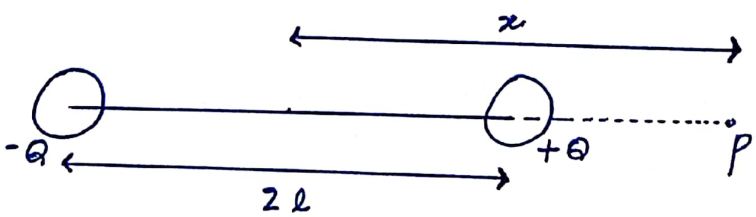
$$\vec{p} = Q(2l)$$

$$\therefore \; E_{net} = \frac{2K\vec{p}x}{(x^2-l^2)^2}$$

As, $x >>> l$

$$\Rightarrow \; E_{net} = \frac{2K\vec{p}x}{x^4}$$

$$\boxed{\vec{E}_{net} = \frac{2K\vec{p}}{x^3}.}$$



Again, considering the same setup and calculating potential at point P (axial).

$$V \text{ due to } -Q = V_1 = \frac{-KQ}{(x+l)}$$

$$V \text{ due to } +Q = V_2 = \frac{KQ}{(x-L)}$$

$$V_p = V_1 + V_2$$

$$= \frac{KQ}{(x-l)} - \frac{KQ}{(x+l)}$$

$$= KQ\left[\frac{x+l-x+l}{x^2-l^2}\right]$$

$$V_p = \frac{KQ(2l)}{x^2-l^2}$$

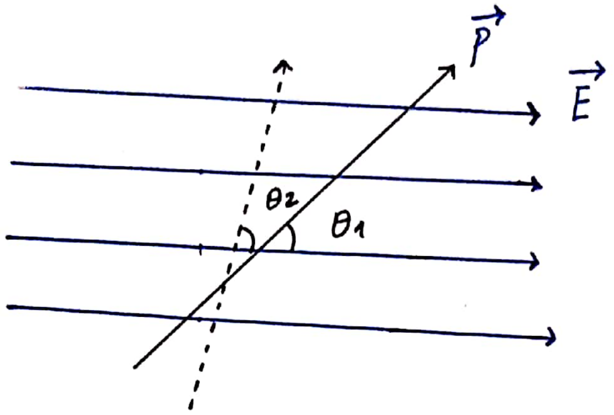
As, $\vec{p} = Q(2l)$

$\therefore \quad V_p = \dfrac{Kp}{x^2-l^2}$

and $\quad x >> l$

$$\Rightarrow \quad \boxed{V_p = \frac{Kp}{x^2}}$$

Q. Write expression for work done in rotating a dipole in an electric field and potential energy.



Let us consider a dipole whose dipole moment is given as '$\vec{p}$', and an uniform electric form $\vec{E}$.

The first configuration of dipole is making an angle 'θ_1', with electric field.

By doing some work, we have changed the configuration, and now dipole is making an angle 'θ_2', with electric field.

Hence,

$$dW = \int \tau \, d\theta,$$

τ is torque experienced by dipole in an electric field

$$\Rightarrow \quad \vec{\tau} = \vec{P} \times \vec{E} = PE\sin\theta$$

$$\therefore \quad \int dW = \int_{\theta_1}^{\theta_2} PE\sin\theta \, d\theta$$

$$W = PE\,[-\cos\theta]_{\theta_1}^{\theta_2}$$

$$\Rightarrow \quad \boxed{W = PE(\cos\theta_1 - \cos\theta_2)}$$

and, we already know

$$W_{\substack{\text{conservative}\\\text{forces}}} = \Delta U$$

Hence $\Delta U = PE(\cos\theta_1 - \cos\theta_2)$

Assuming $\theta_1 = 90°$ and $U_1 = 0$ we get

$$U_2 - U_1 = PE(\cos 90° - \cos\theta_2)$$

$$U_2 = -PE\cos\theta$$

$$\Rightarrow \boxed{U = -\vec{P}\cdot\vec{E}}$$